Chapter 4. Applications of Derivatives

4.2. The Mean Value Theorem

Note. In this section we reveal the relationship between two functions with the same derivative. Probably not surprisingly, $f'(x) = g'(x)$ if and only if $f(x) = g(x) + k$ for some constant k. We prove this using the Mean Value Theorem. We start with a special case of the Mean Value Theorem.

Theorem 4.3. Rolle's Theorem.
Suppose that $y = f(x)$ is continuous at every point of $[a, b]$ and differentiable at every point of (a, b). If $f(a) = f(b) = 0$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Note. The conditions of continuous on $[a, b]$ and differentiable on (a, b) are necessary in Rolle's Theorem, as shown by examples in Figure 4.11. Notice that $f'(c) = 0$ means that the graph of $y = f(x)$ has a horizontal tangent line at $x = c$.

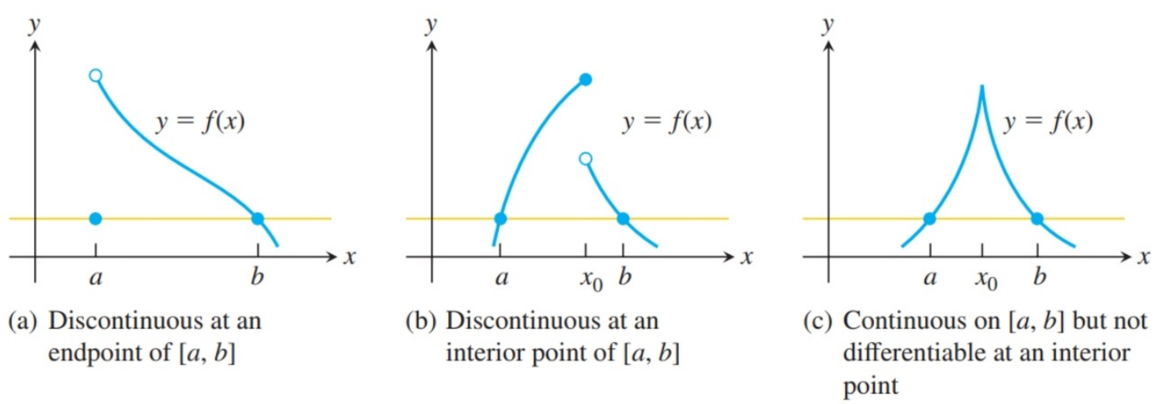

Figure 4.11

Example. Exercise 4.2.60.

Note. We now state the Mean Value Theorem. First, the "mean" part means (!) "average." Informally, the Mean Value Theorem says that for a differentiable function on an interval $[a, b]$, the average rate of change equals the instantaneous rate of change at some point between a and b.

Theorem 4.4. The Mean Value Theorem
Suppose that $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval (a, b). Then there is at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

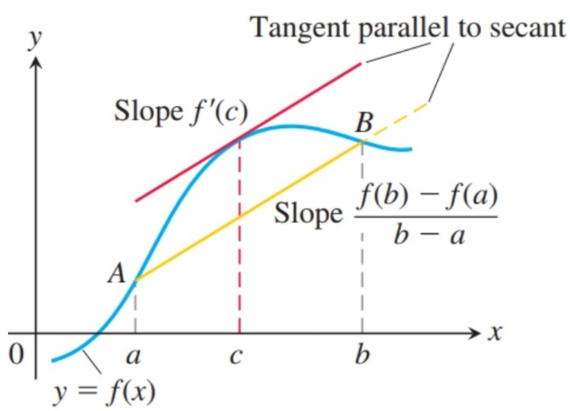

Figure 4.13

Note. Rolle's Theorem is named for Michel Rolle (April 21, 1652 – November 8, 1719), who published the result in 1691 (see the MacTutor History of Mathematics Archive biography of Rolle). Our textbook says that Joseph-Louis Lagrange was the first to state the Mean Value Theorem and Wikipedia says that it was first proved by Augustin Louis Cauchy in 1823 (see the Wikipedia page on the Mean Value Theorem). Proofs of these results are also given in ETSU's senior-graduate level Analysis 1 (MATH 4217/5217); see my online notes on 5.2. Some Mean Value Theorems. These websites were accessed August 13, 2020. We'll also see a Mean Value Theorem for Definite Integrals near the end of Calculus 1 (see Theorem 5.3 of 5.4. The Fundamental Theorem of Calculus). We now illustrate the Mean Value Theorem with some examples.

Examples. Exercise 4.2.2, Exercise 4.2.52, and Exercise 4.2.68.

Note. Recall that derivative of a constant function is 0 (see Theorem 3.3.A). The next result is the *converse* of Theorem 3.3.A.

Corollary 4.1. Functions with Zero Derivatives Are Constant Functions. If $f'(x) = 0$ at each point of an interval I, then $f(x) = k$ for all $x \in I$, where k is a constant.

Note. We next give the relationship between two functions with the same derivative. The result isn't surprising...maybe the surprising part is that we need the Mean Value Theorem to prove the result!

Corollary 4.2. Functions with the Same Derivative Differ by a Constant. If $f'(x) = g'(x)$ at each point of an interval (a, b), then there exists a constant k such that $f(x) = g(x) + k$ for all $x \in (a, b)$.

Example. Exercise 4.2.40.

Example 4.2.A. Finding Velocity and Position from Acceleration. Suppose an object falls vertically in a gravitational field with constant acceleration of -9.8 m/sec^2. If the height at time t is given by $s(t)$ (so that $s''(t) = a(t) = -9.8$ m/sec^2), the initial height is $s(0) = s_0$ m, and the initial velocity is $s'(0) = v(0) = v_0$ m/sec, then find the velocity function $v(t)$ and the height function $s(t)$.

Note. We stated "Algebraic Properties of the Natural Logarithm" in Theorem 1.6.1. The purpose was to use those properties as needed, and to appeal to your past experience with logarithms. We now give proofs of those same properties using the Mean Value Theorem (actually, we use Corollary 4.2) and the differentiation properties of logarithms.

Theorem 1.6.1/Theorem 4.2.A. Algebraic Properties of the Natural Logarithm.

For any numbers $b > 0$ and $x > 0$ we have

1. $\ln bx = \ln b + \ln x$

2. $\ln \frac{b}{x} = \ln b - \ln x$

3. $\ln \frac{1}{x} = -\ln x$

4. $\ln x^r = r \ln x$.

Note. We can use Theorem 1.6.1/Theorem 4.2.B to prove corresponding results for exponential functions.

Theorem 4.2.B. For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} e^{x_2} = e^{x_1 + x_2}$.

2. $e^{-x} = \frac{1}{e^x}$

3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Note. The proof of part (4) of Theorem 4.2.B is similar to the proof of part (1). The proofs of (2) and (3) are to be given in Exercise 4.2.77 and Exercise 4.2.78.

Revised: 9/9/2020