Chapter 4. Applications of Derivatives

4.4. Concavity and Curve Sketching

Note. In this section we use the second derivative of a function to give more details about the shape of the graph of the function.

Note. In the previous section, we used the sign of the derivative to determine when a function is increasing or decreasing. Now we look at the sign of the second derivative. Since the second derivative is the rate of change of the first derivative, and the first derivative gives the slope of a tangent line, then we know that a positive second derivative indicates that the slope of a tangent line is increasing (that is, the first derivative is increasing) and a negative second derivative indicates that the slope of a tangent line is decreasing (that is, the first derivative is decreasing). Figure 4.24 gives a graph of $y = x^3$, along with several tangent lines. Notice that as x increases the slope of tangent lines decreases for $x < 0$, and the slope of tangent lines increases for $x > 0$. That is, for $f(x) = x^3$ we have $f'(x)$ decreasing for $x < 0$ and we have $f'(x)$ increasing for $x > 0$.

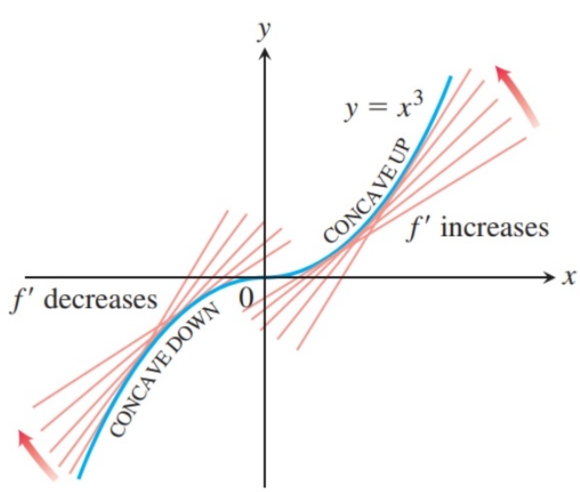


Figure 4.24

Definition. The graph of a differentiable function $y = f(x)$ is

(a) *concave up* on an open interval I if y' is increasing on I

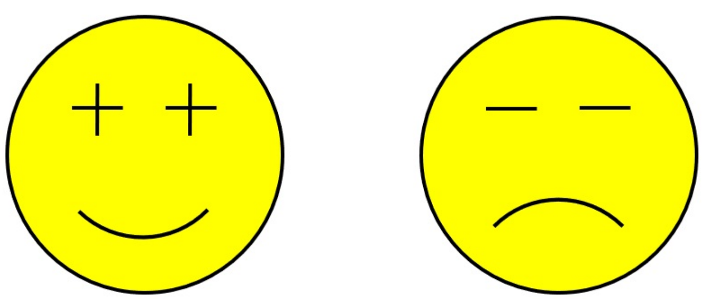
(b) *concave down* on an open interval I if y' is decreasing on I.

A function whose graph is concave up is often called *convex*.

Note. By Corollary 4.3, "The First Derivative Test for Increasing and Decreasing," and the definition of concave up and concave down, we can relate concavity of function f to the sign of f'' as follows.

Theorem 4.4.A. Second Derivative Test for Concavity.
Let $y = f(x)$ be twice-differentiable on an interval I.
1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Note. You can remember the Second Derivative Test for Concavity using the following "concavity emojis":



Note. If f is concave up at point (x_0, y_0), then a tangent line to f at (x_0, y_0) lies **below** the graph of f near (x_0, y_0). If f is concave down at point (x_0, y_0), then a tangent line to f at (x_0, y_0) lies **above** the graph of f near (x_0, y_0).

Definition. A point where the graph of a function has a tangent line and where the concavity changes is a *point of inflection*.

Note. Similar to Theorem 4.2, "Local Extreme Values," if $(c, f(c))$ is a point of inflection of f, then either $f''(c) = 0$ or $f''(c)$ does not exist.

Example. Exercise 4.4.2.

Note. Just as we used the first derivative to find local extrema in Theorem 4.3.A, "First Derivative Test for Local Extrema," we can also use the second derivative to find local extrema under certain conditions.

Theorem 4.5. Second Derivative Test for Local Extrema.
Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

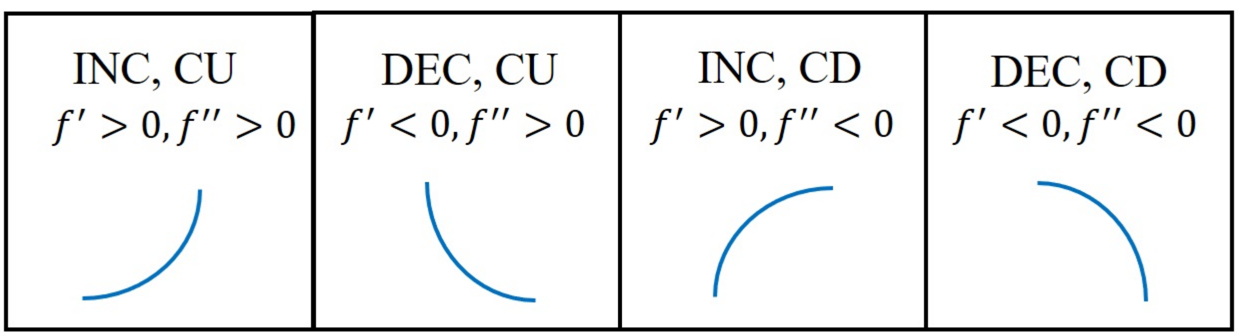
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Example. Exercise 4.4.12.

Note. Notice that Theorem 4.5 only applies to critical points where f' is 0, not to critical points where f' is undefined (since f'' would also be undefined at such a point). Notice that if $f''(c) = 0$ where c is a critical point of f (where $f'(c) = 0$), then Theorem 4.5 does not give any information; it is inconclusive in this case. We'll illustrate the use of Theorem 4.5 after we make some more observations.

Note. If we combine increasing/decreasing information from the first derivative with concavity information from the second derivative, then we see that the graph of a function consists of the following four shapes:



Example. Exercise 4.4.104, Exercise 4.4.42, and Exercise 4.4.54.

Note. We follow these steps in our "Procedure for Graphing $y = f(x)$."

1. Identify the domain of f and any symmetries the curve may have.

2. Find y' and y''.

3. Find the critical points of f, and identify the function's behavior at each one.

4. Find where the curve is increasing and where it is decreasing.

5. Find the points of inflection, if any occur, and determine the concavity of the curve.

6. Identify any asymptotes.

7. Plot key points ("special points"), such as the intercepts and the points found in Steps 3–5, and sketch the curve.

Note. Please read the many good examples of graphing in this section of the book!

Example. Exercise 4.4.74, Exercise 4.4.92, Exercise 4.4.122, and Exercise 4.4.124.

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