Chapter 4. Applications of Derivatives

4.5. Indeterminate Forms and L'Hôpital's Rule

Note. In this section we further study limits. We use derivatives to evaluate limits of certain forms. The technique we use is called L'Hôpital's Rule. According to *Thomas' Calculus*, the result is due to Johann Bernoulli (1667–1748), but was first published by Guillaume François Antoine de l'Hôpital (1661–1704) in the first introductory differential calculus text book.

Johann Bernoulli

G. F. A. de L'Hôpital

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Note. We saw several limits in Chapter 2 which involved quotients of functions in which the limit of the numerator and the limit of the denominator were both 0; we often evaluated these limits with the Factor-Cancel-Substitute method ("FCS"). We now classify limits of this, and a related, type.

Definition. The limit $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is in the

1. $0/0$ *indeterminate form* if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$,

2. ∞/∞ *indeterminate form* if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$.

Note. Beware of the fact that we are not using the symbols "0/0" or "∞/∞" to represent *numbers*, but instead are using them to represent particular types of *limits* (limits may be numbers, limits may be $-\infty$ or ∞ [in which case they are not numbers], or limits may not exist). It is L'Hôpital's Rule that allows us to (sometimes) evaluate these indeterminate forms of limits. We give a proof of it later in the section. First, we state the version of L'Hôpital's Rule which deals with the 0/0 indeterminate form of a limit.

Theorem 4.6. L'Hôpital's Rule.
Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Note. L'Hôpital's Rule also holds for one sided limits, with the appropriate revision of hypotheses. When we apply L'Hôpital's Rule we will put the indeterminate form over the equal sign to indicate that (1) we have used L'Hôpital's Rule, and (2) we have checked to make sure that the limit is in an appropriate indeterminate form.

Example 4.5.A. Use L'Hôpital's Rule to show $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim_{x\to 0}\frac{1-\cos x}{x}$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Example. Exercise 4.5.16 and Exercise 4.5.38.

Note. L'Hôpital's Rule can also be applied to ∞/∞ indeterminate forms. We use reciprocals to rewrite an ∞/∞ indeterminate form as a $0/0$ indeterminate form. As stated in *Thomas' Calculus*, "[m]ore advanced treatments of calculus prove that l'Hôpital's Rule applies to the indeterminate form ∞/∞. . . ." In fact, a proof of this is given in ETSU's Analysis 1 (MATH 4217/5217); see my online notes on 5.2. Some Mean Value Theorems, in particular Theorem 5-11. Specifically, we have the following.

Theorem 4.5.A. L'Hôpital's Rule for ∞/∞ Indeterminate Forms. Suppose that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. If $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$, and $\lim_{x\to a}\frac{f'(x)}{g'(x)} = L$, then then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = L.$$

Note. L'Hôpital's Rule for ∞/∞ indeterminate forms also holds for one sided limits and for the cases $a = \pm\infty$, with the appropriate revision of hypotheses.

Example. Exercise 4.5.46.

Definition. The limit $\lim\limits_{x\to a}(f(x) - g(x))$ is in $\infty - \infty$ *indeterminate form* if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$. The limit $\lim\limits_{x\to a}(f(x)g(x))$ is in $0 \cdot \infty$ *indeterminate form* if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$.

Note. L'Hôpital's Rule can also be applied to $0 \cdot \infty$ indeterminate forms. We simply use reciprocals to rewrite an $0 \cdot \infty$ indeterminate form as with a $0/0$ or ∞/∞ indeterminate form. Similarly, we can rewrite a $0 \cdot \infty$ indeterminate form of a limit as a quotient and potentially apply L'Hôpital's Rule.

Examples. Exercise 4.5.32 and Exercise 4.5.40.

Definition. The limit $\lim\limits_{x\to a} f(x)^{g(x)}$ is in 1^∞ *indeterminate form* if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$. The limit $\lim\limits_{x\to a} f(x)^{g(x)}$ is in 0^0 *indeterminate form* if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. The limit $\lim\limits_{x\to a} f(x)^{g(x)}$ is in ∞^0 *indeterminate form* if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$.

Theorem 4.5.B. If $\lim\limits_{x\to a} \ln f(x) = L$ then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^{\lim_{x\to a} \ln f(x)} = e^L.$$

Here, a may be finite or infinite.

Note. Theorem 4.5.B also holds for one sided limits. This result allows us to extend L'Hôpital's Rule to indeterminate forms 1^∞, 0^0, and ∞^0.

Examples. Exercise 4.5.52, Exercise 4.5.58, and Exercise 4.5.81(b).

Theorem 4.7. Cauchy's Mean Value Theorem.
Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Note. We are now equipped to prove l'Hôpital's Rule (Theorem 4.6). Notice that before the statement of Cauchy's Mean Value Theorem, *Thomas' Calculus* considers a special case of l'Hôpital's Rule "to provide some geometric insight for its reasonableness" (see Figure 4.36).

Revised: 9/16/2020