Chapter 4. Applications of Derivatives

4.7. Newton's Method

Note. In this section we give a numerical iterative technique which can be used to find solutions to the equation $f(x) = 0$ where f is a differentiable function. The technique is called *Newton's Method* or the *Newton-Raphson Method*. This is the only numerical technique we will cover in Calculus 1. You may explore "numerical integration" in Calculus 2 (see Section 8.7). These topics and related ones are covered in the ETSU undergraduate/graduate level class Numerical Analysis (MATH 4257/5257).

Note. In Newton's Method, we try to approximate the solution to $f(x) = 0$ where f is differentiable. We use tangent lines to improve our approximations as follows. First we make an educated guess at what the solution is and call this approximation x_0. We then find the line tangent to $y = f(x)$ at the point $(x_0, f(x_0))$ and follow it to its intersection with the x-axis. This gives an updated approximation x_1 to the solution of $f(x) = 0$. We then iterate this process by using the line tangent to $y = f(x)$ at the point $(x_1, f(x_1))$ and finding its intersection x_2 with the x-axis, and so forth. See Figure 4.48 below.

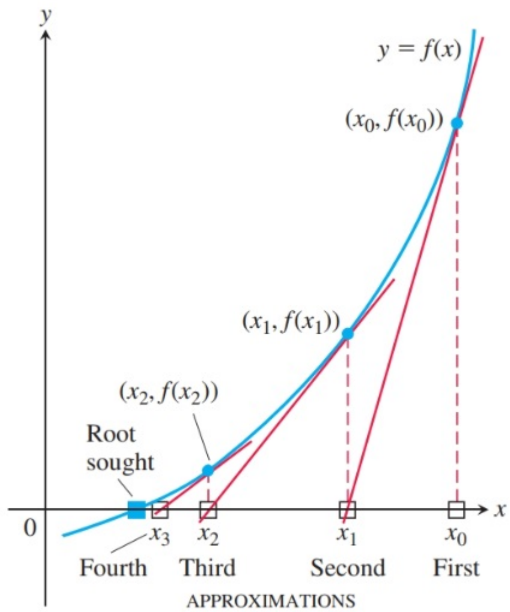

Figure 4.48

Note. The equation of the tangent line to $y = f(x)$ at $(x_n, f(x_n))$ is $y - f(x_n) = f'(x_n)(x - x_n)$. With the x-intercept as the point $(x_{n+1}, 0)$ we then have $0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$ or $x_{n+1} - x_n = -f(x_n)/f'(x_n)$ or $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. See Figure 4.49.

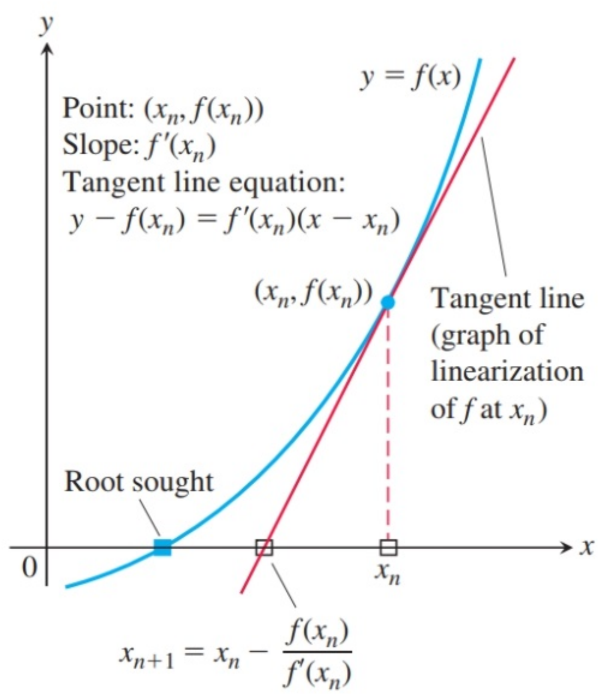

Figure 4.49

Note. The steps for Newton's Method are:

1. Guess a first approximation to a solution of the equation $f(x) = 0$.

2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ if } f'(x_n) \neq 0.$$

Examples. Exercise 4.7.2 and Exercise 4.7.16.

Note. Newton's Method is not guaranteed to produce a solution to $f(x) = 0$. For one thing, there may not be a real solution, so before looking for a solution make sure that one exists! Even when a solution exists, things can go wrong. The method can get "stuck" and oscillate between two values. In Figure 4.53 we see a case where Newton's Method produces x_1 from x_0 and produces x_0 from x_1. So the technique just bounces alternatively between x_0 and x_1; the technique doesn't *converge* to a solution but simply oscillates between these two values. An example of such a function and initial value x_0 is given Exercise 4.7.13.

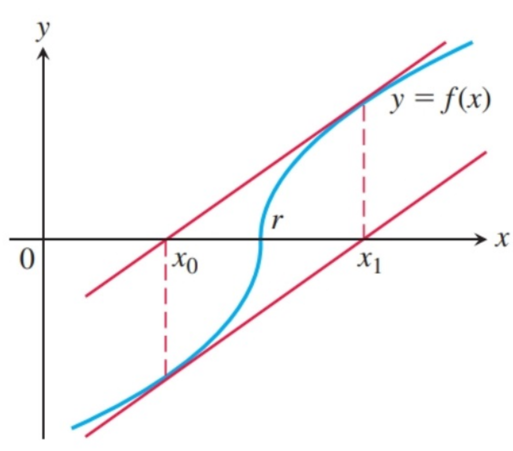

Figure 4.53

Note. Another concern is that Newton's Method may lead us to one solution to $f(x) = 0$, when we had in mind a different solution. This is normally caused by a poor choice of the first guess x_0. Figure 4.54 shows two examples of functions and initial guesses which yield this type of behavior. This leads to a discussion of "basins of attraction" where initial guesses from different basins of attraction lead to different solutions. Also notice that Newton's Method only works when none of the generated x_n have $f'(x_n) = 0$. *If* $f(x_n) = 0$ then the formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ gives division by 0; geometrically, we would be looking for the x-intercept of a line with slope 0 (which does not exist, unless the line is $y = 0$ in which case there are infinitely many x-intercepts). Think of this situation as the technique blowing up!

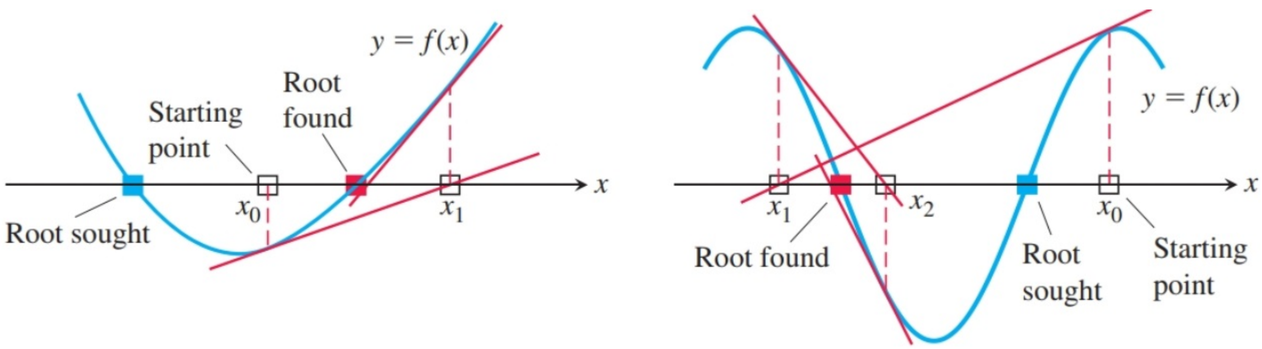

Figure 4.54

Examples. Exercise 4.7.30 and Exercise 4.7.20.

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