# **Chapter 4.** Applications of Derivatives

## **4.8.** Antiderivatives

**Note.** In this section we introduce a process by which we start with the derivative of a function and then we try to find the function itself (or at least the *form* of the function). Many applications require this process. For example, in physics we might be given the acceleration of an object and then try to work "backward" to the velocity and then the position. This "backward" process is called "antidifferentiation."

**Definition.** A function $F$ is an *antiderivative* of a function $f$ if $F'(x) = f(x)$ for all $x$ in the domain of $f$. The most general antiderivative (which is really the **set** of all antiderivatives) of $f$ is the *indefinite integral* of $f$ with respect to $x$, denoted by $\int f(x)\,dx$. The symbol $\int$ is an *integral sign.* The function $f$ is the *integrand* of the integral, and $x$ is the *variable of integration.*

**Note.** By Corollary 4.2, "Functions with the Same Derivative Differ by a Constant" (a corollary to The Mean Value Theorem, Theorem 4.4) we see that any two antiderivatives of the same function differ by a constant. So the indefinite integral of a given function $f$ can be expressed in the form $F(x) + C$ where the symbols "$+C$" are shorthand for the set of all antiderivatives of $f$:

$$\int f(x)\,dx = \{F(x) \mid F'(x) = f(x) \text{ for all } x \text{ in the domain of } f\} = F(x) + C.$$

We say that $C$ is an "arbitrary constant." In this class, we will use "$k$" to represent a specific (fixed) constant. We then have the following.

**Theorem 4.8.** If $F$ is an antiderivative of $f$ on an interval $I$, then the indefinite integral of $f$ on $I$ is $\displaystyle\int f(x)\,dx = F(x) + C$, where $C$ is an arbitrary constant.

Example. Example 4.8.2.

**Note.** Example 4.8.2 is an example of a *differential equation* (or "D.E.") called an *initial value problem* (or "I.V.P."). In it, you are given the derivative of the desired function and the value of that function at some point (an "initial value"). To solve the differential equation is to find the function that has the given derivative and takes on the given value at the given "initial" $x$ value. Certain differential equations are discussed in *Thomas' Calculus* in Chapter 9 (First-Order Differential Equations) and in Chapter 17 (Second-Order Differential Equations). A sophomore level class in differential equations is part of the curriculum in the study of math, physics, or engineering. At ETSU, this class is "Differential Equations (MATH 2120) which has the catalog description (in the 2020-21 Undergraduate Catalog): "Offers first order differential equations and applications; second and higher order linear differential equations and applications; Laplace transforms; and systems of differential equations." We explore differential equations in a little more depth below.

**Note.** In terms of the notation of indefinite integrals, we have the following, each of which can be verified by the rules of differentiation. The table on the next page is Table 4.2 and the table on the page after that is simply Table 4.2 with $k = 1$.

**Table 4.2.** Antiderivative formulas, $k$ a nonzero constant.

| **Indefinite Integral** | **Derivative Formula** |
| --- | --- |
| **1.** $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1,$ | $\displaystyle\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] = x^n$ |
| **2.** $\displaystyle\int \sin kx\,dx = -\frac{\cos kx}{k} + C$ | $\displaystyle\frac{d}{dx}\left[-\frac{\cos kx}{k}\right] = \sin kx$ |
| **3.** $\displaystyle\int \cos kx\,dx = \frac{\sin kx}{k} + C$ | $\displaystyle\frac{d}{dx}\left[\frac{\sin kx}{k}\right] = \cos kx$ |
| **4.** $\displaystyle\int \sec^2 kx\,dx = \frac{1}{k}\tan kx + C$ | $\displaystyle\frac{d}{dx}[\tan kx] = k\sec^2 kx$ |
| **5.** $\displaystyle\int \csc^2 kx\,dx = -\frac{1}{k}\cot kx + C$ | $\displaystyle\frac{d}{dx}[-\cot kx] = k\csc^2 kx$ |
| **6.** $\displaystyle\int \sec kx\tan kx\,dx = \frac{1}{k}\sec kx + C$ | $\displaystyle\frac{d}{dx}[\sec kx] = k\sec kx\tan kx$ |
| **7.** $\displaystyle\int \csc kx\cot kx\,dx = -\frac{1}{k}\csc kx + C$ | $\displaystyle\frac{d}{dx}[-\csc kx] = k\csc kx\cot kx$ |
| **8.** $\displaystyle\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$ | $\displaystyle\frac{d}{dx}\left[\frac{1}{k}e^{kx}\right] = e^{kx}$ |
| **9.** $\displaystyle\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C,\ x \neq 0$ | $\displaystyle\frac{d}{dx}[\ln\lvert x\rvert] = \frac{1}{x}$ (see Example 3.8.3(c)) |
| **10.** $\displaystyle\int \frac{1}{\sqrt{1-k^2x^2}}\,dx = \frac{1}{k}\sin^{-1}kx + C$ | $\displaystyle\frac{d}{dx}\left[\frac{1}{k}\sin^{-1}kx\right] = \frac{1}{\sqrt{1-k^2x^2}}$ |
| **11.** $\displaystyle\int \frac{1}{1+k^2x^2}\,dx = \frac{1}{k}\tan^{-1}kx + C$ | $\displaystyle\frac{d}{dx}\left[\frac{1}{k}\tan^{-1}kx\right] = \frac{1}{1+k^2x^2}$ |
| **12.** $\displaystyle\int \frac{1}{x\sqrt{k^2x^2-1}}\,dx = \sec^{-1}kx + C,\ kx > 1$ | $\displaystyle\frac{d}{dx}[\sec^{-1}kx] = \frac{1}{x\sqrt{k^2x^2-1}}$ |
| **13.** $\displaystyle\int a^{kx}\,dx = \left(\frac{1}{k\ln a}\right)a^{kx} + C,\ a > 0,\ a \neq 1$ | $\displaystyle\frac{d}{dx}[a^{kx}] = a^{kx}k\ln a$ |

**Table 4.2.A.** Table 4.2 with $k = 1$.

| **Indefinite Integral** | **Derivative Formula** |
|---|---|
| **1.** $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C, n \neq -1,$ | $\dfrac{d}{dx}\left[\dfrac{x^{n+1}}{n+1}\right] = x^n$ |
| **2.** $\int \sin x\,dx = -\cos x + C$ | $\dfrac{d}{dx}[-\cos x] = \sin x$ |
| **3.** $\int \cos x\,dx = \sin x + C$ | $\dfrac{d}{dx}[\sin x] = \cos x$ |
| **4.** $\int \sec^2 x\,dx = \tan x + C$ | $\dfrac{d}{dx}[\tan x] = \sec^2 x$ |
| **5.** $\int \csc^2 x\,dx = -\cot x + C$ | $\dfrac{d}{dx}[-\cot x] = \csc^2 x$ |
| **6.** $\int \sec x \tan x\,dx = \sec x + C$ | $\dfrac{d}{dx}[\sec x] = \sec x \tan x$ |
| **7.** $\int \csc x \cot x\,dx = -\csc x + C$ | $\dfrac{d}{dx}[-\csc x] = \csc x \cot x$ |
| **8.** $\int e^x\,dx = e^x + C$ | $\dfrac{d}{dx}[e^x] = e^x$ |
| **9.** $\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C,\ x \neq 0$ | $\dfrac{d}{dx}[\ln\lvert x\rvert] = \dfrac{1}{x}$ |
| **10.** $\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$ | $\dfrac{d}{dx}\left[\sin^{-1} x\right] = \dfrac{1}{\sqrt{1-x^2}}$ |
| **11.** $\int \dfrac{1}{1+x^2}\,dx = \tan^{-1} x + C$ | $\dfrac{d}{dx}\left[\tan^{-1} x\right] = \dfrac{1}{1+x^2}$ |
| **12.** $\int \dfrac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C,\ x > 1$ | $\dfrac{d}{dx}[\sec^{-1} x] = \dfrac{1}{x\sqrt{x^2-1}}$ |
| **13.** $\int a^x\,dx = \left(\dfrac{1}{\ln a}\right) a^x + C,\ a > 0,\ a \neq 1$ | $\dfrac{d}{dx}[a^x] = a^x \ln a$ |

**Note 4.8.A.** Based on the properties of differentiation, we have the following "linearity rules" for indefinite integrals. Suppose $F$ is an antiderivative of $f$, $G$ is an antiderivative of $g$, and $k$ is a constant.

**1.** Constant Multiple Rule:

$$\int kf(x)\,dx = k\int f(x)\,dx = kF(x) + C.$$

**4.** Sum or Difference Rule:

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx = F(x) \pm G(x) + C.$$

This is stated in Table 4.3 in the book.

**Examples.** Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c).

**Examples.** Exercises 4.8.32, 4.8.46, 4.8.52, 4.8,54, 4.8.66, and 4.8.76.

**Definition.** A *differential equation* is an equation relating an unknown function $y$ of $x$ and one or more of its derivatives. A function whose derivatives satisfy a differential equation is called a *solution* of the differential equation and the set of all solutions is called the *general solution*. The problem of finding a specific function $y$ of $x$ which is a solution to a differential equation and satisfies certain *initial condition(s)* of the form $y(x_0) = y_0$, $y'(x_0) = y'_0$, etc., is called an *initial value problem*.

**Examples.** Exercise 4.8.94, Exercise 4.8.102, Exercise 4.8.108.

**Examples.** Exercise 4.8.120 and Exercise 4.8.124.