Chapter 5. Integrals

Note. On the first page of this chapter, the text book states that "A great achievement of classical geometry was obtaining formulas for the areas and volumes of triangle, spheres, and cones." Informally, it is easy to find the area of a region with flat sides (it can be partitioned into triangles and it's easy to find the area of a triangle). The areas of "curvey" things, such as circles and areas bounded by part of a parabola are harder to find (as are volumes of such curvey things as spheres and cones). These areas and volumes were introduced into "classical geometry" largely by Archimedes (287–212 BCE). Euclid (circa 300 BCE) in his *Elements of Geometry*, Book XII, Proposition 2 shows that the area of a circle is proportional to the square of the radius of the circle. The constant π is (by definition) the ratio of the circumference of a circle to the diameter of the circle (so that the circumference of a circle of radius r is $2\pi r$). Archimedes in his *Measurement of a Circle* proved that the area of a circle of radius r is πr^2. He also famously demonstrated in this work that $\pi \approx 22/7$ (he proved that $\frac{22}{7} < \pi < 3\frac{10}{71}$). With the area of a circle established, it is then easy to find the volume of a right circular cylinder (in this case, $V = \pi r^2 h$ where r is the radius and h is the height). Euclid states in Book XII, Proposition 10 of his *Elements* that the volume of a cone is 1/3 of the volume of a cylinder of the same radius and height (implying the volume of a right circular cone of base radius r and height h is $V = \frac{1}{3}\pi r^2 h$).

Archimedes in *On the Sphere and Cylinder* also proves that for a sphere of radius r, the surface area is $S = 4\pi r^2$ and the volume is $V = \frac{4}{3}\pi r^3$. Euclid uses the "method of exhaustion" to establish his results; in this one assumes, for example, that the volume of a right circular cone is less than $\frac{1}{3}\pi r^2 h$ to get a contradiction and then assumes the volume is more than $\frac{1}{3}\pi r^2 h$ and gets another contradiction. Archimedes uses a technique similar to what we introduce in this chapter in finding the area bounded by a straight line and a parabola in *The Method (of Mechanical Theorems)*. We will be very capable of finding such areas ourselves, but we have the advantage of over 2,000 years of mathematical advancements at our disposal that Archimedes didn't have!

Euclid (circa 325–265 BCE)

Archimedes (287–212 BCE)

Images from MacTutor History of Mathematics Archive.

For more details, see my online presentation "Archimedes: 2,000 Year Ahead of His Time," the impressive website of David Joyce Euclid's Elements online, and *The Works of Archimedes* on GoogleBooks (each accessed 8/16/2020).

5.1. Area and Estimating with Finite Sums

Note. In this section we introduce a technique by which we approximate the area under a curve by partitioning the area into rectangles (the area of each rectangle is easy to find) and adding up the areas of the rectangles. We make the estimation precise in the next section and find a shortcut to calculating these areas which involve antiderivatives in the third section of this chapter.

Example 5.1.A. We estimate the area under $y = 1 - x^2$ and above the x-axis for $x \in [0, 1]$. See Figure 5.1

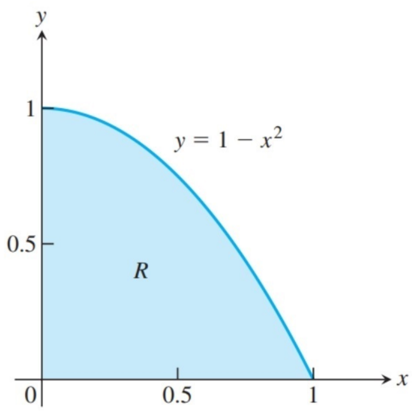


Figure 5.1

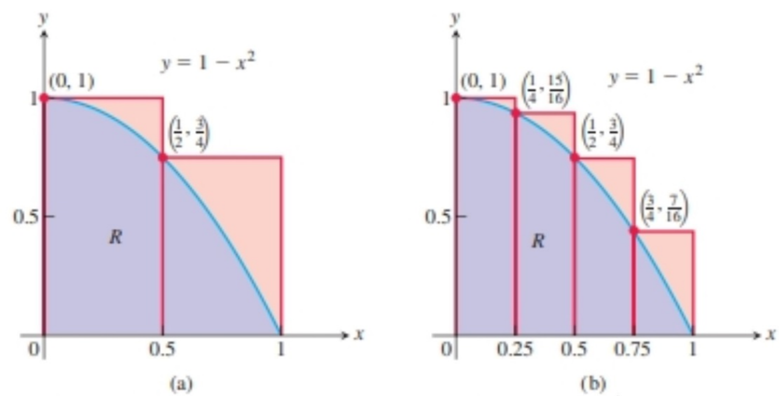


Figure 5.2

First, we cut the region into two parts of equal width and then introduce rectangles with heights determined from function values. See Figure 5.2(a). We use the left hand endpoints of the two intervals on the x-axis which are determined by cutting the region. We then have that the sum of the areas of the resulting two rectangles is $A = (1)(0.5) + \left(\frac{3}{4}\right)(0.5) = \frac{7}{8} = 0.875$. We see from Figure 5.2(a) that this is an overestimation of the desired area.

Second, we cut the region into four parts of equal width and again introduce rectangles with heights determined from function values. See Figure 5.2(b). We use the left hand endpoints of the four intervals on the x-axis which are determined by cutting the region. We then have that the sum of the areas of the resulting four rectangles is $A = (1)(0.25) + \left(\frac{15}{16}\right)(0.25) + \left(\frac{3}{4}\right)(0.25) + \left(\frac{7}{16}\right)(0.25) = \frac{25}{32} = 0.78125.$

We see from Figure 5.2(b) that this is an overestimation of the desired area, but a better estimate than when we cut the region into only two parts. Since both of these estimations of the area are based on the maximum function value over each little interval, these are called *upper sums.*

We could also use the minimum function value over each little interval to determine the heights of rectangles, resulting in *lower sums.* When the region is cut into four parts of equal width this leads to the sum of the areas or the rectangles: $A = \left(\frac{15}{16}\right)(0.25) + \left(\frac{3}{4}\right)(0.25) + \left(\frac{7}{16}\right)(0.25) + (0)(0.25) = \frac{17}{32} = 0.53125$. See Figure 5.3(a). Notice that we now have upper and lower bounds on the actual desired area; we have that it is between $\frac{17}{32} = 0.53125$ and $\frac{25}{32} = 0.78125$.

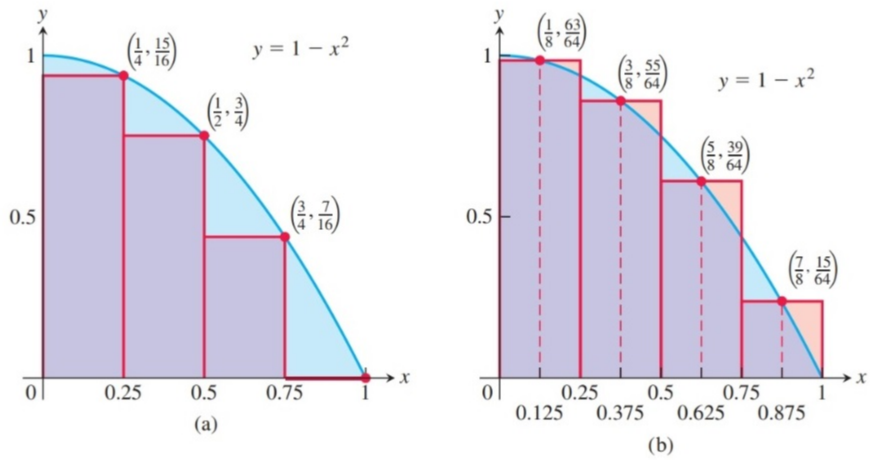


Figure 5.3

Next, we try something that should give a better approximation. We use the midpoint of each little interval to determine the heights of rectangles. See Figure 5.3(b). This leads to the sum of the areas of the rectangles:

$$A = \left(\frac{63}{64}\right)(0.25) + \left(\frac{55}{64}\right)(0.25) + \left(\frac{39}{64}\right)(0.25) + \left(\frac{15}{64}\right)(0.25) = \frac{43}{64} = 0.671875.$$

This technique is called the *midpoint rule.*

Note. In each of the approximations above, the interval $[a, b]$ was divided into n subintervals of equal length $\Delta x = (b-a)/n$, and f was evaluated at a point in each subinterval, so $x = c_k$ was chosen from the kth subinterval and then $f(c_k)$ was used as a height of a rectangle of areas $f(c_k)\Delta x$ (the terms "height" and "area" imply that each $f(c_k) \geq 0$, which we had above). Each of the above sums were then of the form:

$$f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \cdots + f(c_k)\Delta x + \cdots + f(c_{n-1})\Delta x + f(c_n)\Delta x.$$

Notice that the smaller the width of the rectangles (in this case, given by making the number n of rectangles larger), the better we expect the approximation to the exact area to be. In Figure 5.4 we have $n = 16$; in (a) we use lower sums and in (b) we use upper sums. In this case, the lower sum gives an area of 0.634765625 and the upper sum gives an area of 0.697265625 (the midpoint rule for $n = 16$ gives an area of 0.6669921875. We will see in the next section that the actual area of the region is $R = 2/3$.

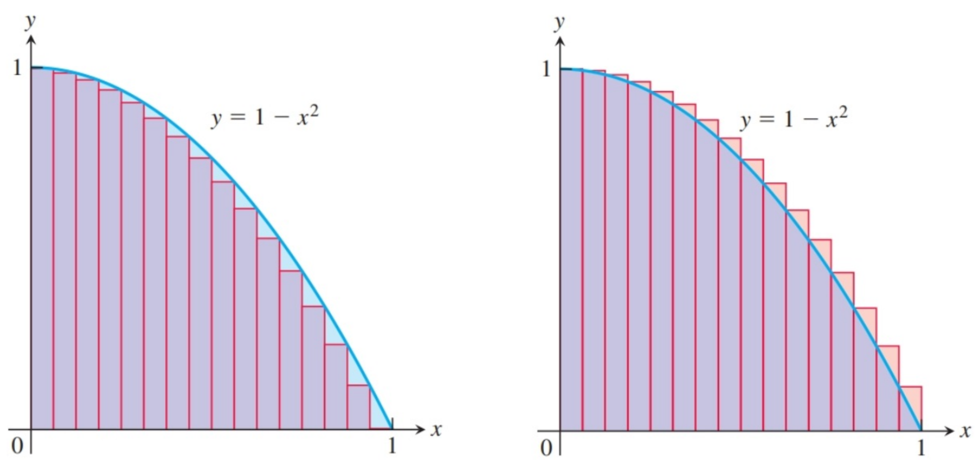


Figure 5.4

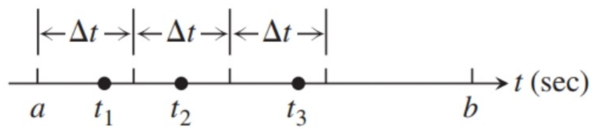
Note. In Table 5.1, the lower sum, upper sum, and midpoint sum are given for the number of subintervals n as 2, 4, 16, 50, 100, and 1000. Notice as n gets larger each of the types of sum gets closer to $R = 2/3$. So we expect to find the exact area by taking a limit of some of these sums(!).

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.53125	0.671875	0.78125
16	0.634765625	0.6669921875	0.697265625
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

Example. Exercise 5.1.6.

Note. We now consider a similar example to those above, but we consider a velocity function $v(t)$ of (as the book says) a car that moves in a straight line. Since the velocity of the car changes with time, we take little "slices of time" (like the subintervals above) and pick a time t_k in the kth little slice of time to estimate the velocity on that slice. With the slices of time each of length Δt, the distance traveled in the kth slice of time is estimated as $v(t_k)\Delta t$ (think "distance = rate $\times$ time"). If the time interval is $[a, b]$, then we visualize the slices/subintervals and the t_k's as:



If we now sum the distances over all of the subintervals, we get an approximation for the net distance traveled as:

$$D \approx v(t_1)\Delta t + v(t_2)\Delta t + v(t_3)\Delta t + \cdots + v(t_k)\Delta t + \cdots + v(t_{n-1})\Delta t + v(t_n)\Delta t.$$

Since velocity can be negative (and the "distance" that results from some of the products make be negative; this corresponds to the car backing up, say), then the approximation is to the *displacement* (that is, the final position minus the initial position) instead of the total distance traveled. If we wished to find the total distance traveled, we would replace the velocity function $v(t)$ with the speed function $|v(t)|$ and the approximation to the *total distance traveled* is then:

$$|v(t_1)|\Delta t + |v(t_2)|\Delta t + |v(t_3)|\Delta t + \cdots + |v(t_k)|\Delta t + \cdots + |v(t_{n-1})|\Delta t + |v(t_n)|\Delta t.$$

Example. Exercise 5.1.10.

Note. We know how to average a collection of numbers and now we want to introduce an idea of how we might find the average of a continuous function f on an interval $[a, b]$. We do so using the area under the function. In Figure 5.6(b) we have the graph of continuous function g on the interval $[a, b]$. We want to find a value c such that the "area" of a rectangle with base $b - a$ (the length of the interval $[a, b]$) and "height" c (shown in Figure 5.6(b)) is the same as the "area" under the the graph of g. We then define this value c as the *average value of g on* $[a, b]$. We use quote marks since these terms are only used for positive quantities and, since g could be negative, these quantities could be negative here. We can approximate the "area" under the graph as above, and then approximate c as this "area" divided by $b - a$.

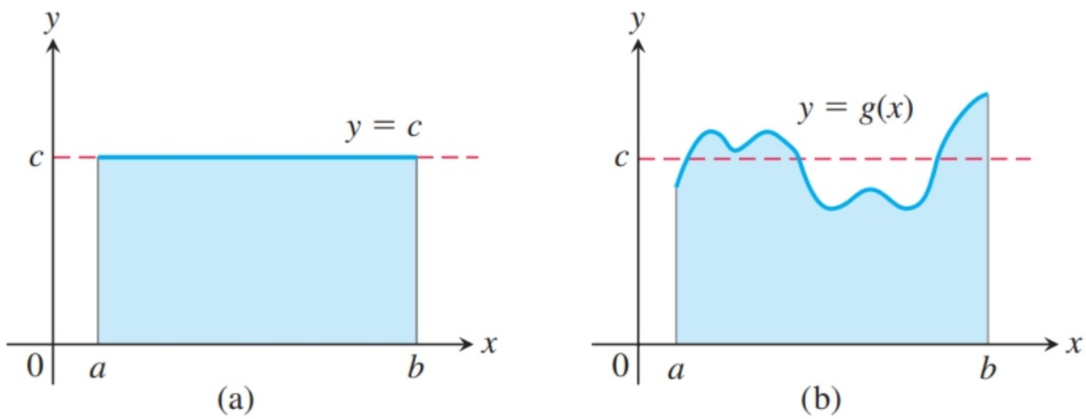


Figure 5.6

Example. Exercise 5.1.16.

Example. Exercise 5.1.20.

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