Chapter 5. Integrals

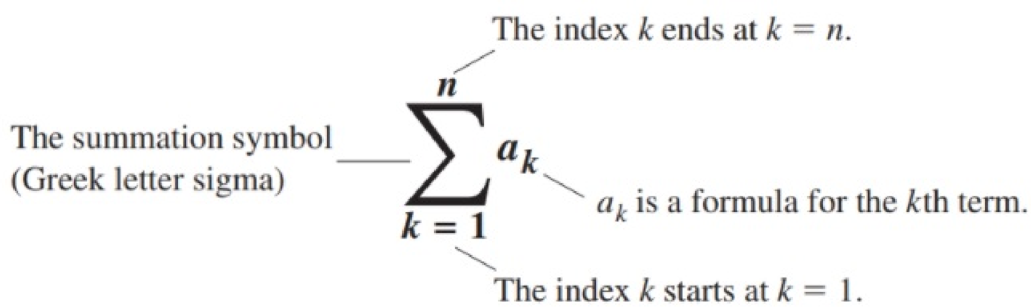
5.2. Sigma Notation and Limits of Finite Sums

Note. In this section we introduce a shorthand notation for summation. We will use this summation notation in the next section when we define the *exact* area under a curve.

Note. We use the *sigma notation* to denote sums:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The Greek letter Σ ("sigma," corresponding to our letter "S") stands for "sum." The *index of summation* k reflects where the sum begins and ends, and in general a_k is some function of k which gives the kth *term* of the sum:



Examples. Exercise 5.2.2 and Exercise 5.2.12.

Note. In Appendix A.2. Mathematical Induction, the following are established (see Exercise A.2.11).

Theorem 5.2.A. Algebra for Finite Sums.

1. *Sum Rule*: $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. *Difference Rule*: $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. *Constant Multiple Rule*: $\displaystyle\sum_{k=1}^{n} c a_k = c\sum_{k=1}^{n} a_k$

4. *Constant Value Rule*: $\displaystyle\sum_{k=1}^{n} c = nc$

Example. Exercise 5.2.18.

Note. In Appendix A.2. Mathematical Induction, the following are established (see Example A.2.5, Exercise A.2.9, and Exercise A.2.10).

Theorem 5.2.B. The Sum of Powers of the First n Natural Numbers.

1. The first n natural numbers: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

2. The first n natural numbers squared: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. The first n natural numbers cubed: $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$

Examples. Exercise 5.2.24 and Exercise 5.2.28.

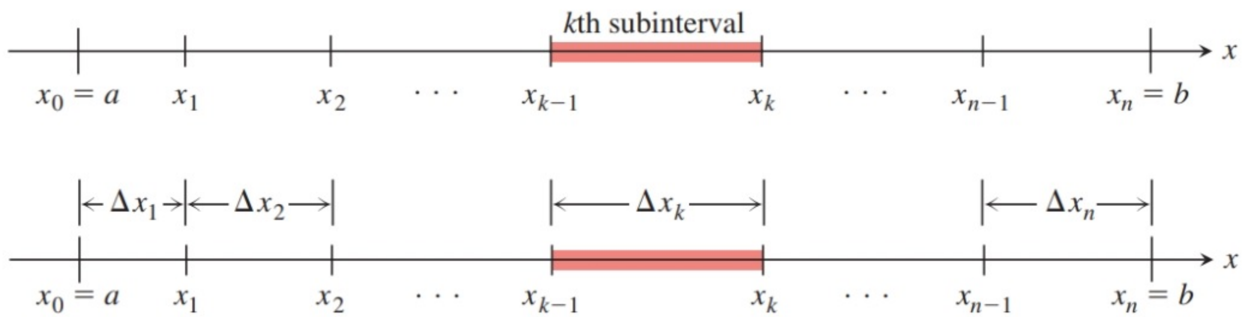
Definition. A *partition* of the interval $[a, b]$ is a set

$$P = \{x_0, x_1, \ldots, x_n\} \text{ where } a = x_0 < x_1 < \cdots < x_n = b.$$

Partition P determines n closed *subintervals*

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

The length of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$.



Note. We now estimate the area bounded between a function $y = f(x)$ and the x-axis. We make the convention that the area bounded **above** the x-axis and below the function is **positive**, and the area bounded **below** the x-axis and above the curve is **negative**. We estimate this "area" by choosing a $c_k \in [x_{k-1}, x_k]$ and we use $f(c_k)$ as the "height" of a rectangle with base $[x_{k-1}, x_k]$. Then a partition P of $[a, b]$ can be used to estimate this "area" by adding up the "area" of these rectangles. See Figure 5.9 below.

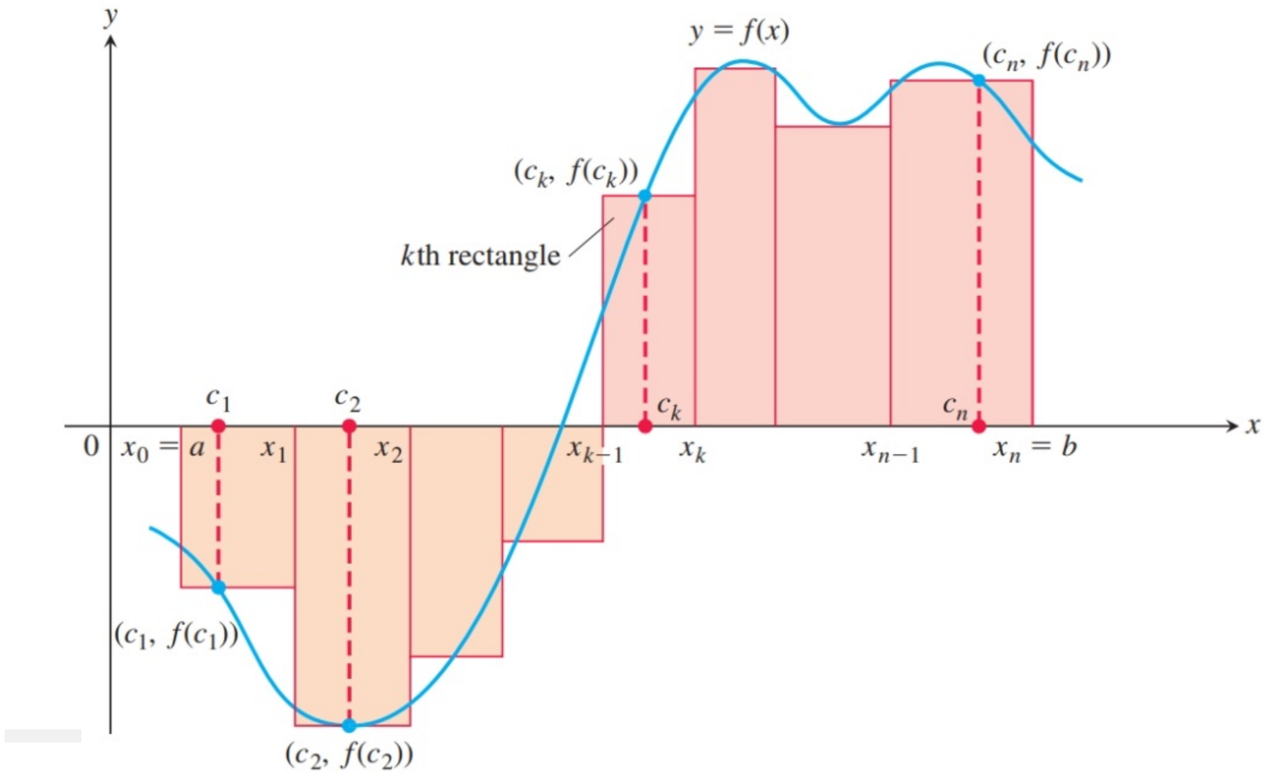


Figure 5.9

Definition. With the above notation, a *Riemann sum of f on the interval* $[a, b]$ is a sum of the form

$$s_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k.$$

Example. Exercise 5.2.38.

Example 5.2.5. Partition the interval $[0, 1]$ into n subintervals of the same width, give the lower sum approximation of area under $y = 1 - x^2$ based on n, and find the limit as $n \to \infty$ (in which case the width of the subintervals approaches 0).

Definition. The *norm* of a partition $P = \{x_0, x_1, \ldots, x_n\}$ of interval $[a, b]$, denoted $\|P\|$, is length of the largest subinterval:

$$\|P\| = \max_{1 \le k \le n} \Delta x_k = \max_{1 \le k \le n} (x_k - x_{k-1}).$$

Note. If $\|P\|$ is "small," then a Riemann sum is a "good" approximation of the "area" described above.

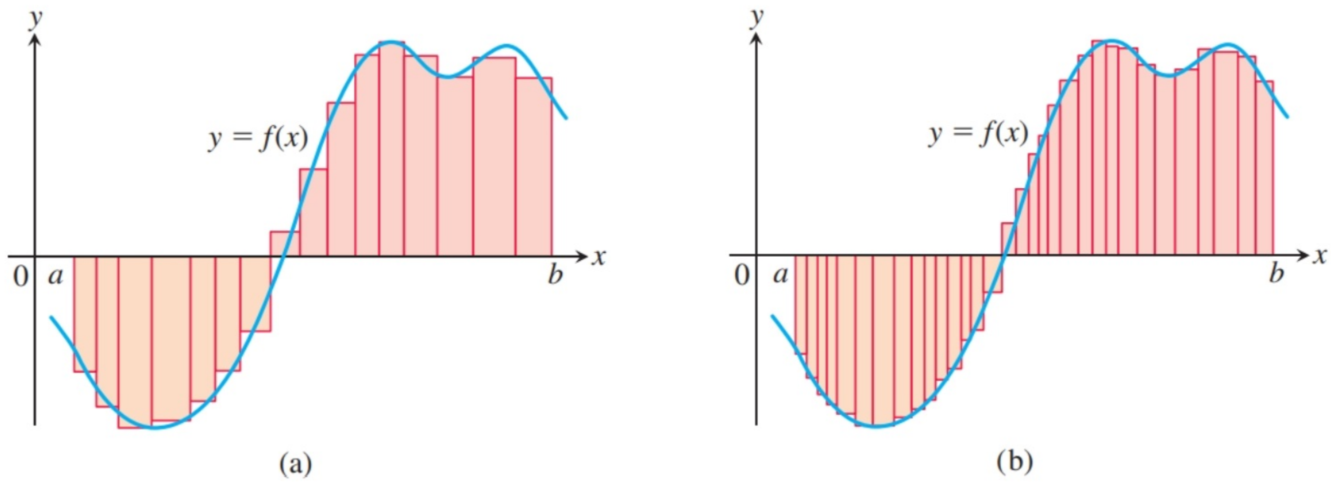


Figure 5.10

Note. If $[a, b]$ is partitioned into n subintervals of equal length, then that length is $\Delta x_k = \Delta x = (b - a)/n$. In this case, if $n \to \infty$ then $\|P\| \to 0$.

Example. Exercise 5.2.48.

Revised: 10/10/2020