Chapter 5. Integrals

5.3. The Definite Integral

Note. In this section we define the definite integral of a function f over interval $[a, b]$. We state properties of the definite integral (in Theorem 5.2), give several examples, and define the average value of a function. In the next section we give an easy way to calculate definite integrals involving antiderivatives.

Definition. Let f be a function defined on a closed interval $[a, b]$. We say that a number J is the *definite integral of f over* $[a, b]$ and that J is the limit of the Riemann sums if the following condition is satisfied: Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of $c_k \in [x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n} f(c_k)\,\Delta x_k - J\right| < \varepsilon.$$

We denote $J = \displaystyle\int_a^b f(x)\,dx$ and say that f is *integrable* on $[a, b]$.

Note. If we compare the roles of $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ and J with a function, say F, of the partition P and a limit L in the formal definition of limit in 2.3. The Precise Definition of a Limit, and compare the values $\|P\|$ and 0 to x and c, then we see the correspondence

$$0 < |x - c| < \delta \text{ implies } |F(x) - L| < \varepsilon$$

with

$$0 < |\|P\| - 0| < \delta \text{ implies } \left|\sum_{k=1}^{n} f(c_k)\,\Delta x_k - J\right| < \varepsilon.$$

Since the first statement is the definition of $\lim_{x\to c} f(x) = L$, then the second statement corresponds to $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = J$. Therefore we have

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx.$$

Example. Exercise 5.3.6.

Note. When we deal with applications of integration, we will often think of definite integrals as sums. Notice, however, that strictly speaking they are not sums, but they are *limits* of sums.

Note. We have now introduced three ideas, each different from the other, but each related to the other (as we will see when we state the Fundamental Theorem of Calculus). We have:

Name of Object	**Type of Object**
ANTIDERIVATIVE	*FUNCTION*
INDEFINITE INTEGRAL	*COLLECTION* or *SET*
DEFINITE INTEGRAL	*NUMBER*

Antiderivatives and indefinite integrals are related by the fact that the indefinite integral of a function f is the set of all antiderivatives of f. The Fundamental Theorem of Calculus, to be seen in the next section, will relate antiderivatives and definite integrals (and therefore will relate definite and indefinite integrals).

Note 5.3.A. In the previous section we saw examples of limits of Riemann sums where each subinterval was the same size. Such a partition of $[a,b]$ is often called a "regular partition," but our text book refers to this as "equal-width subintervals." If the definite integral exists (and we will soon see that if f is continuous on $[a,b]$ then the integral exists) then we can calculate it using an equal-width partition of $[a,b]$ with n subintervals by letting $n \to \infty$ for associated Riemann sums. In so doing, we partition $[a,b]$ into n subintervals, each of length $\Delta x = (b-a)/n$. With the endpoints of the subintervals as $a = x_0, x_1, \ldots, x_k, x_{k+1}, \ldots, x_n = b$ we have $x_k = a + k(b-a)/n$, $c_k \in [x_{k-1}, x_k]$, and the Riemann sum $\displaystyle\sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right)$. Then the value of the Riemann integral is

$$J = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right).$$

Notice that the partition has norm $\|P\| = (b-a)/n$ and so as $n \to \infty$ then $\|P\| \to 0$. If we choose $c_k \in [x_{k-1}, x_k]$ as $c_k = x_k = a + k(b-a)/n$ (the right endpoint of each subinterval) then this leads to the value of the Riemann integral as:

$$J = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right).$$

Note. The next result gives a large class of functions (which includes continuous functions) that are Riemann integrable. The idea behind the integrability of continuous functions is given in Exercises 5.3.86 and 5.3.87. In Additional and Advanced Exercises 5.11 to 5.18 (at the end of Chapter 5) it is argued that "piecewise-continuous functions" (which is a function continuous on $[a,b]$, except for a finite number of jump discontinuities) are also integrable, and this class is considered in the theorem as well.

Theorem 5.1. Integrability of Continuous Functions.
If a function f is continuous on an interval $[a,b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a,b]$.

Note. A rigorous proof of Theorem 5.1 for continuous functions requires the notion of "uniform continuity." This idea is introduced in Exercise 5.3.87. For the details of a proof, see my online notes for Analysis 2 (MATH 4227/5227) on 6.1. The Riemann Integral or The Riemann-Lebesgue Theorem (see Theorem 6-7).

Note. Not all functions are integrable. It turns out that a bounded function on $[a,b]$ is integrable if it is not "too badly discontinuous." This is quite a long story and is spelled out in detail in The Riemann-Lebesgue Theorem mentioned in the previous note. The next example gives a non-integrable function.

Example 5.3.1. Show that the function $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ is not Riemann integrable over the interval $[0,1]$.

Note. We now give rules of definite integrals. The first two rules are in fact definitions. We give proofs of most of the remaining claims.

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

1. *Order of Integration*: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (this in fact is a definition)

2. *Zero Width Interval*: $\int_a^a f(x)\,dx = 0$ (this too is a definition)

3. *Constant Multiple*: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

4. *Sum and Difference*: $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity*: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality*: If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination*: $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$

Note. Figure 5.11 gives illustrations of Theorem 5.2 parts 2–7:

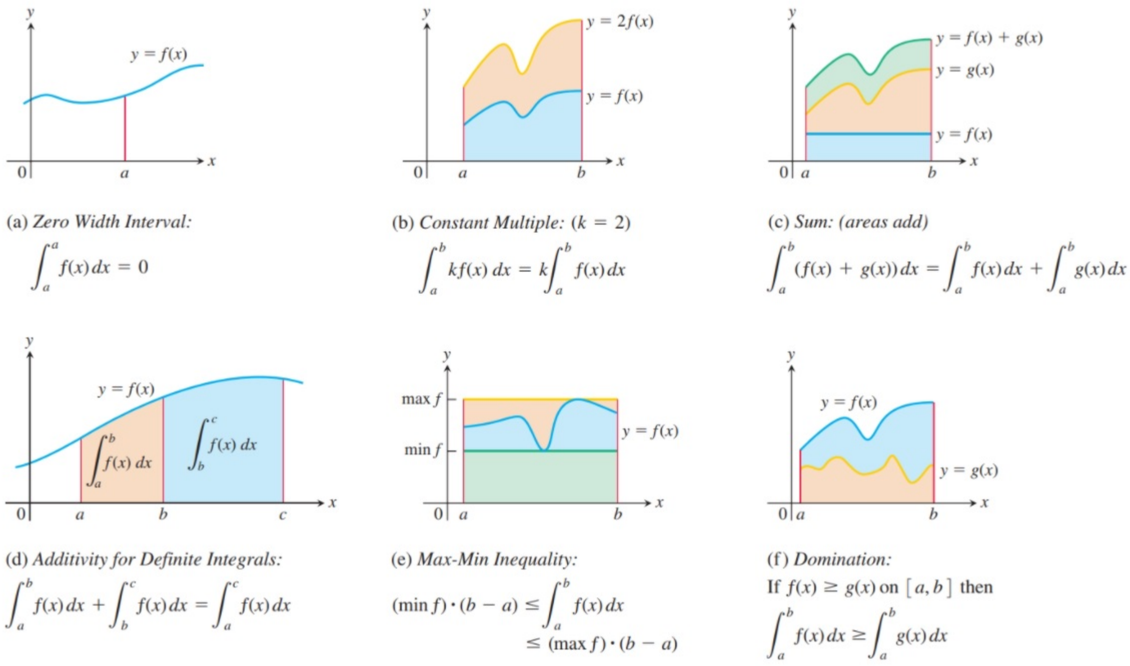

Figure 5.11

Example. Exercise 5.3.10.

Example. Exercise 5.3.63: Let c be a constant. Prove that $\displaystyle\int_a^b c\,dx = c(b-a)$. This is the text book's equation (3).

Example 5.3.A. Use a regular partition of $[a, b]$ with $c_k = x_k$ to prove that for $a < b$: $\displaystyle\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$. This is the text book's equation (2).

Example. Exercise 5.3.65: Use a regular partition of $[a, b]$ with $c_k = x_k$ to prove that for $a < b$: $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$. This is the text book's equation (4).

Example. Exercise 5.3.36.

Definition. If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the *area under the curve* $y = f(x)$ *from* a *to* b is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

Example. Exercise 5.3.18.

Definition. If f is integrable on $[a, b]$, then its *average (mean) value* over $[a, b]$ is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example. Exercise 5.3.62.

Examples. Exercise 5.3.76 and Exercise 5.3.88.

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