Chapter 5. Integrals

5.4. The Fundamental Theorem of Calculus

Note. In this section we relate the value of a definite integral to an antiderivative of the integrand. This is accomplished in the Fundamental Theorem of Calculus, Part 2 (Theorem 5.4(b)). With this result, we no longer need to express definite integrals as limits of Riemann sums, as we did in the previous section. This makes the evaluation of definite integrals "easy" (well, if finding antiderivatives is easy... this is a conversation that will continued in Calculus 2 [MATH 1920]).

Note. Recall that the Mean Value Theorem, Theorem 4.4, states that, for a differentiable function on an interval, the average ("mean") rate of change equals the instantaneous rate of change at some point. The next theorem is similar, but states that a continuous function on an interval attains its average value at some point. Recall that the average ("mean") value of function f on interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Theorem 5.3. The Mean Value Theorem for Definite Integrals.
If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

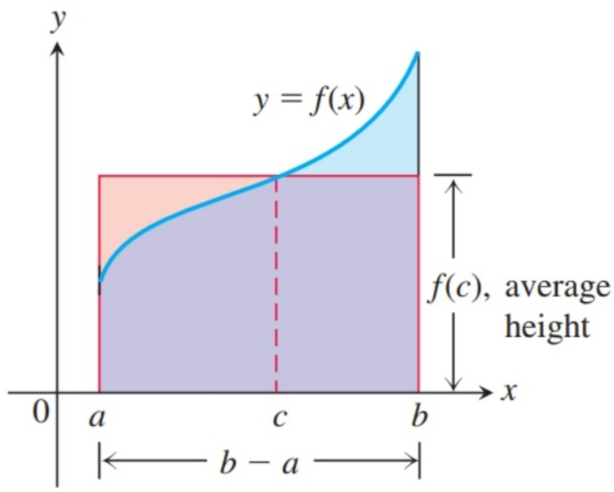

Figure 5.16

Example. Example 5.4.1.

Note. We now motivate the Fundamental Theorem of Calculus, Part 1. Suppose f is an integrable function over a finite interval I. Then for any $a \in I$ and $x \in I$, $x \neq a$, we can define a new function $F(x) = \int_a^x f(t)\,dt$. We then have that $F(x)$ represents the signed area between $y = f(t)$ and the t-axis over the interval $[a, x]$:

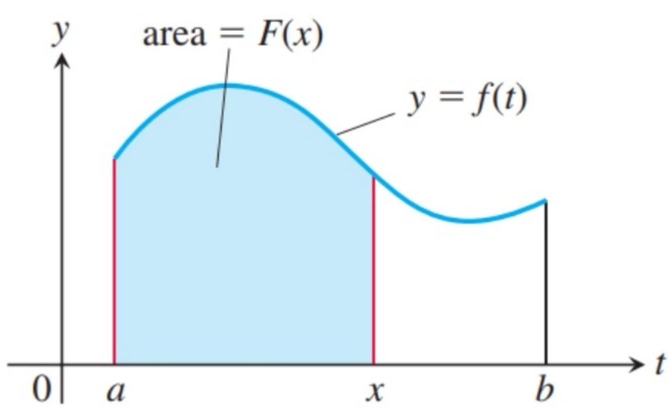

Figure 5.19

Now suppose we consider a difference quotient for F: $\dfrac{F(x+h)-F(x)}{h}$. Now $F(x+h)-F(x)$ is the signed area between $y = f(t)$ and the t-axis over the interval $[x, x+h]$. This areas is approximately a rectangle of signed height $f(x)$ and signed width h:

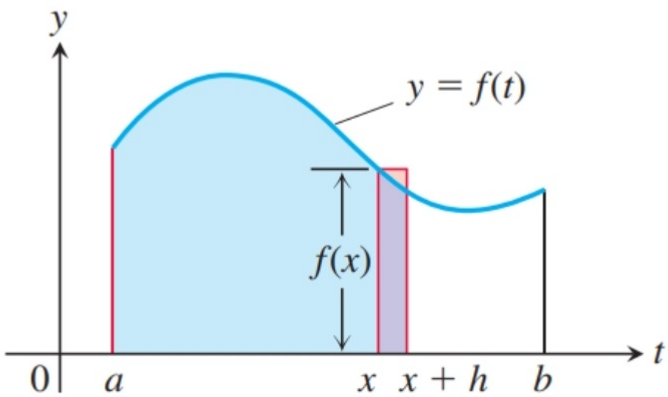

Figure 5.20

So we have $F(x+h)-F(x) \approx hf(x)$. Therefore the difference quotient satisfies $\dfrac{F(x+h)-F(x)}{h} \approx \dfrac{hf(x)}{h} = f(x)$. If f is continuous, then as h approaches 0 this approximation gets better. So we claim (to be formally justified in the proof of The Fundamental Theorem of Calculus, Part 1) that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x).$$

Theorem 5.4(a). The Fundamental Theorem of Calculus, Part 1.

If f is continuous on $[a, b]$ then the function

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point x in $[a, b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

Examples. Exercise 5.4.46, Exercise 5.4.48, and Exercise 5.4.54.

Theorem 5.4(b). The Fundamental Theorem of Calculus, Part 2.

If f is continuous at every point of $[a,b]$ and if F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Examples. Exercise 5.4.6, Exercise 5.4.14, Exercise 5.4.22, Exercise 5.4.64.

Example. Exercise 5.4.82: Find the linearization of $g(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt$ at $x = -1$.

Examples. Exercise 5.4.72 and Exercise 5.4.74.

Example. Example 5.4.8.

Note. Notice that the net change of a function F over the interval $[a,b]$ is $F(b) - F(a)$. So by The Fundamental Theorem of Calculus, Part 2 (Theorem 5.4(b)) we have the following.

Theorem 5. The Net Change Theorem.

The net change in a function $F(x)$ over an interval $a \leq x \leq b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\,dx.$$