Chapter 5. Integrals

5.5. Indefinite Integrals and the Substitution Method

Note. In this section we introduce the first technique of integration (called "u-substitution"). This is the only technique introduced in Calculus 1, but in Calculus 2 (MATH 1920) you will see several more; the title of Chapter 8 is "Techniques of Integration."

Note. Suppose we denote a function $f(x)$ as u: $u = f(x)$. Then, for n a real number, $n \neq -1$, by the Power Rule and Chain Rule for differentiation,

$$\frac{d}{dx}\left[\frac{u^{n+1}}{n+1}\right] = \frac{d}{dx}\left[\frac{(f(x))^{n+1}}{n+1}\right] = (f(x))^{\overset{\curvearrowright}{n}}[f'(x)].$$

As an indefinite integral, we can write this as

$$\int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + C.$$

Since the differential of u is $du = f'(x)\,dx$, we write

$$\int u^n\,du = \int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C.$$

Example. Exercise 5.5.20.

Note. The next result is a generalization of the observation made above. It describes the technique commonly called "u-substitution." It is effectively "the Chain Rule backward."

Theorem 5.6. The Substitution Rule. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Note. The text book describes the substitution method applied to $\int f(g(x))g'(x)\,dx$ step-wise as:

1. Substitute $u = g(x)$ and $du = (du/dx)\,dx = g'(x)\,dx$ to obtain $\int f(u)\,du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

Notice that in the first step, *all* x's in the original integral must be converted to u's (in one way or another) in the new integral.

Examples. Exercise 5.5.6, Exercise 5.5.32, Exercise 5.5.56, and Exercise 5.5.60.

Examples. Example 5.5.7(c): Evaluate $\int \tan x\,dx$. Example 5.5.8(b): Evaluate $\int \sec x\,dx$.

Note. Similar to the previous two examples, we can also integrate $\cot x$ and $\csc x$ (see Exercises 5.5.71 and 5.5.72). We then get the following indefinite integrals (expanding the list we had in Table 4.2 from Section 4.8):

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x\,dx = -\ln|\csc x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

Notice that the book states $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$, but this equals the value given above. We prefer the version given above because this then follows the pattern we have had in the setting of both derivatives and antiderivatives of having sines and cosines "together," secants and tangents "together," and cosecants and cotangents "together."

Examples. Exercise 5.5.68 and Exercise 5.5.78.

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