Chapter 5. Integrals

5.6. Substitution and Area Between Curves

Note. In this section we combine the technique of u-substitution introduced in the previous section with the Fundamental Theorem of Calculus, Part 2 (Theorem 5.4(b)) to evaluate definite integrals. We then define the area between the graphs of two functions in terms of definite integrals.

Theorem 5.7. Substitution in Definite Integrals.
If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Note. We can use u-substitution in definite integrals:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du$$

where $u = g(x)$, and $du = g'(x)\,dx$.

Examples. Exercise 5.6.22 and Exercise 5.6.18.

Note. Recall that a function f is *even* if $f(-x) = f(x)$ for all x in the domain of f. A function f is *odd* if $f(-x) = -f(x)$ for all x in the domain of f. Integration of such functions over symmetric intervals (that is, intervals of the form $[-a, a]$) can be simplified as follows.

Theorem 5.8. Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example. Exercise 5.6.14.

Note 5.6.A. Consider the area bounded above by $y = f(x)$ and below by $y = g(x)$ for $a \le x \le b$ (where $f(x) \ge g(x)$ for $x \in [a, b]$), as shown in Figure 5.25. As in the definition of definite integral, we partition the interval $[a, b]$ as $P = \{a = x_0, x_1, \ldots, x_n = b\}$ and consider the resulting rectangles with bases $(x_k - x_{k-1})$ and heights $(f(c_k) - g(c_k))$ where $c_k \in [x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$. So the area of the kth rectangle is $\Delta A_k = (f(c_k) - g(c_k))\Delta x_k$ where $\Delta x_k = (x_k - x_{k-1})$ for $k = 1, 2, \ldots, n$. We then approximate the area between the curves as $A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n}(f(c_k) - g(c_k))\Delta x_k$. When we vary the partitions P of $[a, b]$ and let $\|P\| \to 0$, then we expect the area A to equal

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n}(f(c_k) - g(c_k))\Delta x_k = \int_a^b (f(x) - g(x))\,dx.$$

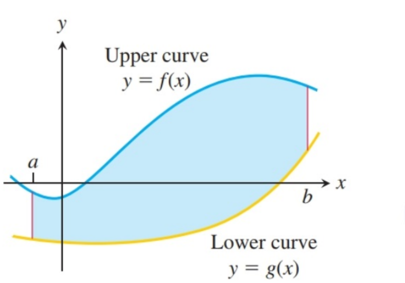


FIGURE 5.25 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

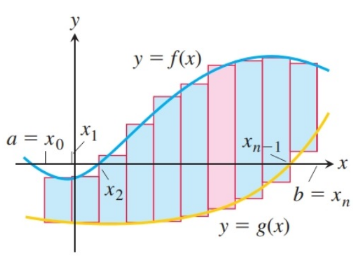


FIGURE 5.26 We approximate the region with rectangles perpendicular to the x-axis.

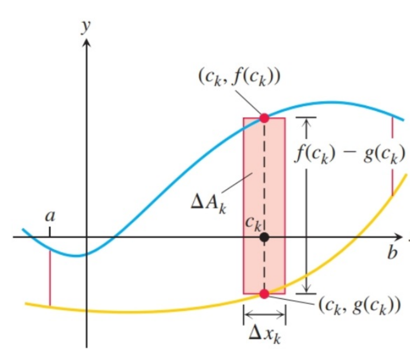


FIGURE 5.27 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

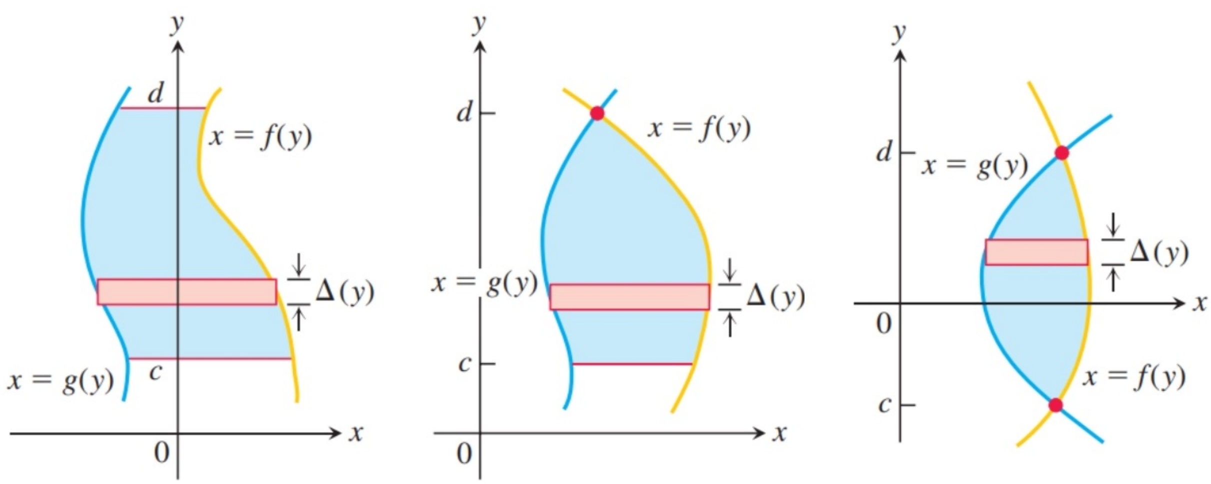
Definition. If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the *area of the region between the curves* $y = f(x)$ *and* $y = g(x)$ *from* a *to* b is the integral of $(f - g)$ from a to b:

$$A = \int_a^b (f(x) - g(x))\,dx.$$

Example. Exercise 5.6.58.

Examples. Example 5.6.6, Exercise 5.6.62, and Exercise 5.6.90.

Note. We can also find the area bounded to the right by $x = f(y)$ and bounded on the left by $x = g(y)$ for $c \leq y \leq d$ (where $f(y) \geq g(y)$ for $y \in [c, d]$), as shown in the figure below (left), using integrals. If the curves intersect and one or both endpoints of $[c, d]$ then (as shown in the figure below, center and right) we can still use integrals to fund the area; in particular, in the figure on the right we find the area bounded by the two curves. In each case, the area is $A = \int_c^d (f(y) - g(y))\,dy$.



Example. Exercise 5.6.78.

Note 5.6.B. When considering areas bounded between functions as considered above, we take a heuristic shortcut to considering partitions P of $[a, b]$ and limits of sums as $\|P\| \to 0$ as described in Note 5.6.A above by taking "dx" slices (as opposed to Δx slices) and "integrating them up" (as opposed to adding them up and then taking a limit). This is not mathematically rigorous, but it is a shortcut to estimating areas between curves that works! This shortcut will be particularly useful in Chapter 6, "Applications of Definite Integrals," in which you will do things to the little slices (such as revolving them around an axis to generate a "washer" or "shell" as is done in 6.1. Volumes Using Cross-Sections and Section 6.2. Volumes Using Cylindrical Shells).

Examples. Exercise 5.6.108, Exercise 5.6.114, and Exercise 5.6.118.

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