Chapter 3. Derivatives

3.3. Differentiation Rules

Note. In this section we streamline the computation of derivatives by establishing rules of differentiation that will allow us to quickly compute derivatives of complicated functions. We state (and prove) the rules as theorems.

Note. If we think of a derivative as a rate of change, then we would expect the derivative of a constant function to be 0, which it is as we now show.

Theorem 3.3.A. Derivative of a Constant Function.
If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}[c] = 0.$$

Note. Some of the examples we saw in the previous two sections foreshadowed the following. Speaking of examples, we will state several rules for differentiation before working some examples.

Theorem 3.3.B. Derivative Power Rule for Positive Integers.
If n is a positive integer, then

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Note. Theorem 3.3.B is to be interpreted as $\frac{d}{dx}[x] = 1$, even though with $n = 1$ we have $nx^{n-1} = 1x^0$ which is 1 except at 0 where it is undefined (since "0^0" is not defined).

Note. The Power Rule for Positive Integers actually holds for any real number. We have not yet defined what it means to have an irrational exponent, but we will do so when we explore exponential functions in more detail. At that time (in Section 3.8) we will give a proof of the following. For now, we accept it as true.

Theorem 3.3.C. Derivative Power Rule (General Version).
If n is any real number, then

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

for all x where the powers x^n and x^{n-1} are defined.

Note. Given that derivatives are defined in terms of limits, it is not surprising that some of the rules of differentiation are similar to rules of limits. This is the case for the following two results.

Theorem 3.3.D. Derivative Constant Multiple Rule
If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}[cu] = c\frac{du}{dx}.$$

Theorem 3.3.E. Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u+v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}[u+v] = \frac{du}{dx} + \frac{dv}{dx}.$$

Note. Combining Theorems 3.3.D and 3.3.E, we have that for constants c_1 and c_2, and for differentiable u and v that

$$\frac{d}{dx}[c_1u(x)+c_2v(x)] = c_1\frac{d}{dx}[u(x)] + c_2\frac{d}{dx}[v(x)] = c_1u'(x)+c_2v'(x).$$

You may have the class Linear Algebra (MATH 2010) in your future; the quantity $c_1u(x)+c_2v(x)$ is a "linear combination" or u and v and for this reason differentiation is called a *linear operator*. This allows to show the following, which is left as a homework exercise.

Exercise 3.3.73. If $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, then $P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$.

Examples. Exercises 3.3.4(a), Exercise 3.3.12(a), Exercise 3.3.34, and Example 3.3.4.

Note. We now differentiate an exponential function $f(x) = a^x$ where $a > 0$. By definition,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} a^x \frac{a^h - 1}{h} \\ &= a^x \lim_{h\to 0} \frac{a^h - 1}{h}. \end{aligned}$$

We **claim without justification** that $\lim_{h\to 0} \frac{a^h - 1}{h}$ exists and is some number L_a dependent on a. (For a clean discussion of this result, see sections 7.2 and 7.3 of *Thomas Calculus*, Standard 11th Edition—notes are available online at 7.2. Natural Logarithms and 7.3. The Exponential Functions. There is a version of this in the 14th Early Transcendentals edition in "Section 7.1. The Logarithm Defined as an Integral".) With $x = 0$, we have $f'(0) = a^0 \lim_{h\to 0} \frac{a^h - 1}{h} = \lim_{h\to 0} \frac{a^h - 1}{h} = L_a$. We will see the precise value of L_a in Section 3.8. Derivatives of Inverse Functions and Logarithms (it is $L_a = \ln a$). Now $f'(0)$ is the slope of the graph of $y = a^x$ at $x = 0$. Motivated by Figure 3.13, we see that there is a value of a somewhere between 2 and 3 such that this slope is 0.

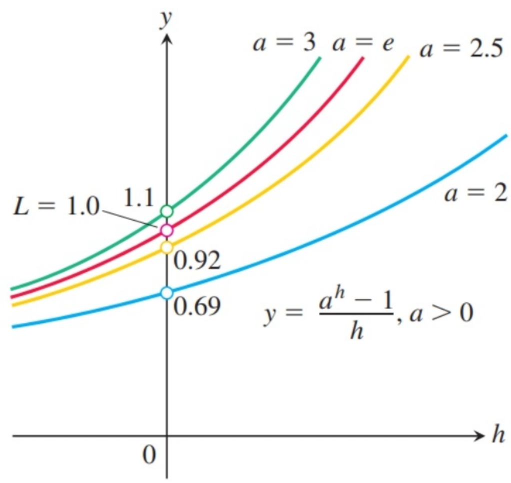

Figure 3.13

We define e to be the number for which the slope of the line tangent to $y = e^x$ is $m = 1$ at $x = 0$. That is, we define e such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. One can determine numerically (for a technique, see Section 3.8. Derivatives of Inverse Functions and Logarithms) that $e \approx 2.71828182845904 59$. What is *natural* about the natural exponential function e^x is a *calculus* property—a *differentiation* property.

Theorem 3.3.F. Derivative of the Natural Exponential Function.

$$\frac{d}{dx}[e^x] = e^x.$$

Example 3.3.A. Differentiate $f(x) = x + 5e^x$.

Note. We now state two very useful rules. With these two rules in place, we are almost done developing the rules of differentiation.

Theorem 3.3.G. Derivative Product Rule.

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx} = [u'](v) + (u)[v'].$$

Example 3.3.B. Differentiate $f(x) = (4x^3 - 5x^2 + 4)(7x^2 - x)$.

Theorem 3.3.H. Derivative Quotient Rule.

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} = \frac{[u'](v) - (u)[v']}{(v)^2}.$$

Example. Exercise 3.3.20.

Note. In the previous two examples, we used a square bracket notation that can be employed to make applications of the Derivative Product Rule and the Derivative Quotient Rule simple fill-in-the-blank problems. In Example 3.3.B we had:

$$\frac{d}{dx}[(4x^3 - 5x^2 + 4)(7x^2 - x)] = [12x^2 - 10x](7x^2 - x) + (4x^3 - 5x^2 + 4)[14x - 1].$$

In Exercise 3.3.20 we had

$$\frac{d}{dt}\left[\frac{t^2 - 1}{t^2 + t - 2}\right] = \frac{[2t](t^2 + t + 2) - (t^2 - 1)[2t + 1]}{(t^2 + t - 2)^2}.$$

This suggests my "square bracket notation" that can be used to draw a picture of both the Derivative Product and Derivative Quotient Rules. We draw the Derivative Product Rule (Theorem 3.3.G) as:

$$\frac{d}{dx}[(\quad)(\quad)] = [\quad](\quad) + (\quad)[\quad].$$

We use the parentheses to represent quantities that have not been differentiated and we use the square brackets to represent quantities that have been differentiated, as required by the Derivative Product Rule. Similarly, we draw the Derivative Quotient Rule (Theorem 3.3.H) as:

$$\frac{d}{dx}\left[\frac{(\ \ )}{(\ \ )}\right] = \frac{[\ \ ](\ \ ) - (\ \ )[\ \ ]}{(\ \ )^2}.$$

Again, the parentheses contain quantities that have not been differentiated and the square brackets contain quantities that have been differentiated, as required by the Derivative Quotient Rule. In both cases, you need to know what goes where, but this notation helps with the "bookkeeping" needed in a complicated differentiation problem, especially one that involves multiple applications of the Derivative Product and Derivative Quotient Rules. **A word of warning:** The square brackets do not mean "take the derivative of the quantity inside," but instead mean that the quantity inside *is* the derivative of some part of the original given function! The symbols "$\frac{d}{dx}[\ \ ]$" are used to represent that a quantity is meant to be differentiated. We will also use this notation when we differentiate compositions of functions in Section 3.6. The Chain Rule. For more details on my square bracket notation, see R. Gardner, A Useful Notation for Rules of Differentiation, published in *The College Journal of Mathematics*, **24**(4) (1993) 351-352, and reprinted in *The Calculus Collection: A Resource for AP and Beyond*, pages 257-58, edited by C. Diefenderfer and R. Nelsen, The Mathematical Association of America, 2010.

Exercise 3.3.48. Differentiate $f(x) = \dfrac{(x^2+x)(x^2-x+1)}{x^4}$.

Example 3.3.C. Differentiate $f(x) = \dfrac{(5x^3 - 4x + 2)(3x^2 + 7x - 8)}{(6x^5 - 9x^2 + 2x)(e^x)}$.

Solution. We treat this as a quotient of products. Since there are two products (one in the numerator and one in the denominator), then we will have to use the Derivative Product Rule twice. We have

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\frac{(5x^3 - 4x + 2)(3x^2 + 7x - 8)}{(6x^5 - 9x^2 + 2x)(e^x)}\right] \\
&= \frac{\frac{d}{dx}[(5x^3 - 4x + 2)(3x^2 + 7x - 8)]\left((6x^5 - 9x^2 + 2x)(e^x)\right) - \left((5x^3 - 4x + 2)(3x^2 + 7x - 8)\right)\frac{d}{dx}\left[(6x^5 - 9x^2 + 2x)(e^x)\right]}{\left((6x^5 - 9x^2 + 2x)(e^x)\right)^2} \\
&= \frac{\frac{d}{dx}[(5x^3 - 4x + 2)(3x^2 + 7x - 8)]\left((6x^5 - 9x^2 + 2x)(e^x)\right)}{\left((6x^5 - 9x^2 + 2x)(e^x)\right)^2} - \frac{\left((5x^3 - 4x + 2)(3x^2 + 7x - 8)\right)\frac{d}{dx}\left[(6x^5 - 9x^2 + 2x)(e^x)\right]}{\left((6x^5 - 9x^2 + 2x)(e^x)\right)^2} \\
&= \frac{[[15x^2 - 4](3x^2 + 7x - 8) + (5x^3 - 4x + 2)[6x + 7]]((6x^5 - 9x^2 + 2x)(e^x))}{((6x^5 - 9x^2 + 2x)(e^x))^{\mathbf{2}}} \\
&\quad - \frac{((5x^3 - 4x + 2)(3x^2 + 7x - 8))[[30x^4 - 18x + 2](e^x) + (6x^5 - 9x^2 + 2x)[e^x]]}{((6x^5 - 9x^2 + 2x)(e^x))^{\mathbf{2}}}.
\end{aligned}$$

Here we have used red brackets to illustrate the two uses of the product rule, and used blue brackets to illustrate the one use of the quotient rule (though it is split up over two lines).

Examples. Exercise 3.3.78 and Exercise 3.3.77(a,b).

Note. We see from Exercise 3.3.77 that for differentiable functions u_1, u_2, u_3, u_4 we have

$$\frac{d}{dx}[u_1 u_2 u_3] = [u_1'](u_2)(u_3) + (u_1)[u_2'](u_3) + (u_1)(u_2)[u_3']$$

and

$$\frac{d}{dx}[u_1 u_2 u_3 u_4] = [u_1'](u_2)(u_3)(u_4)+(u_1)[u_2'](u_3)(u_4)+(u_1)(u_2)[u_3'](u_4)+(u_1)(u_2)(u_3)[u_4'].$$

The pattern is pretty clear from the square bracket notation, and we could use Mathematical Induction (see Appendix A.2 and Example A.2.B) to prove to show that the pattern holds for the product of any number of differentiable functions. Namely, to differentiate a product of n differentiable functions, $u_1, u_2, u_3, \ldots, u_{n-1}, u_n$, we add together n copies of the product $u_1 u_2 u_3 \cdots u_{n-1} u_n$, but we differentiate u_1 in the first product, differentiate u_2 in the second product, and so forth up to the last (the nth) product. In terms of the square bracket notation *per se*, we have:

$$\begin{aligned}\frac{d}{dx}[(\ \ )(\ \ )(\ \ )\cdots(\ \ )(\ \ )] \;=\;& \;[\ \ ](\ \ )(\ \ )\cdots(\ \ )(\ \ )\\ &+(\ \ )[\ \ ](\ \ )\cdots(\ \ )(\ \ )\\ &+(\ \ )(\ \ )[\ \ ]\cdots(\ \ )(\ \ )+\cdots\\ &+(\ \ )(\ \ )(\ \ )\cdots[\ \ ](\ \ )\\ &+(\ \ )(\ \ )(\ \ )\cdots(\ \ )[\ \ ].\end{aligned}$$

Note. If f is a differentiable function with derivative f', then we could potentially differentiate f' to find the *second derivative*, f'', of f. With $y = f(x)$ we have the notations

$$f''(x) = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2} = y''.$$

We can similarly calculate higher *order* derivatives:

$$y''' = y^{(3)} = \frac{d}{dx}[y''],\ y^{(4)} = \frac{d}{dx}[y'''], \ldots, y^{(n)} = \frac{d}{dx}[y^{(n-1)}].$$

Examples. Exercise 3.3.42, Exercise 3.3.66, and Exercise 3.3.80.

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