Chapter 4. Applications of Derivatives

4.3. Monotonic Functions and the First Derivative Test

Note. In this section we use the first derivative of a function to determine where the function is increasing or decreasing and where it has extrema. This information is then used to graph a function. We start with a definition from 1.1. Functions and Their Graphs.

Definition. Let f be a function defined on an interval I. Then

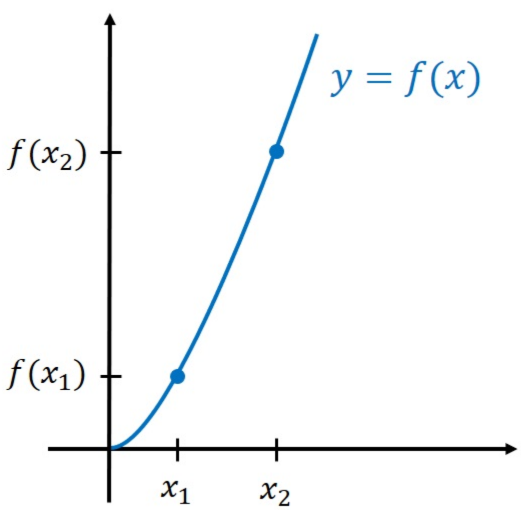
1. f *increases* on I if for all points x_1 and x_2 in I,

$$\text{if } x_1 < x_2 \text{ then } f(x_1) < f(x_2).$$

2. f *decreases* on I if for all points x_1 and x_2 in I,

$$\text{if } x_1 < x_2 \text{ then } f(x_1) > f(x_2).$$

A function that is increasing or decreasing on I is called *monotonic* on I.



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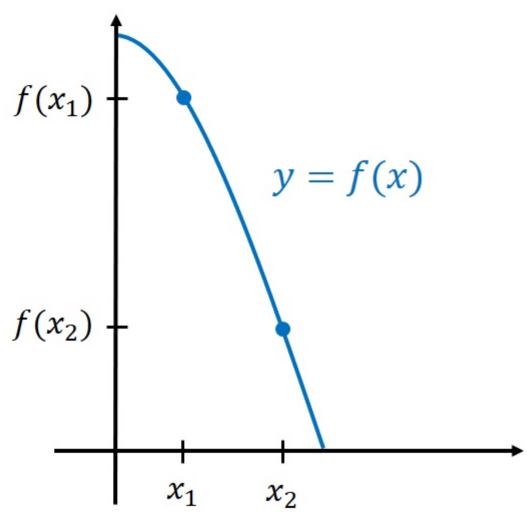

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Note. Not surprisingly, we can relate the increasing and decreasing properties of a function to its derivative.

Corollary 4.3. The First Derivative Test for Increasing and Decreasing. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b)

If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.

If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

Example. Exercise 4.3.28(a): Find the sets on which the function $g(x) = x^4 - 4x^3 + 4x^2$ is increasing and decreasing. Use the critical points of g to make a table of the sign of g' using test values from the intervals on which g' has the same sign.

Note. Figure 4.21 gives an example of the graph of a function which is continuous on $[a, b]$ and which has several critical points in the interval (namely, c_1, c_2, c_3, c_4, and c_5). At some of the critical points the derivative is 0 (at c_1, c_2, c_3, and c_5) and at one of the critical points the derivative is undefined (at c_4). The function is differentiable except at c_4 and the sign of the derivative is indicated between the critical points. To motivate the First Derivative Test For Local Extrema (Theorem 4.3.A), notice where the function has local and absolute extrema on $[a, b]$. At the left endpoint a of the domain, the function has a local minimum since the function increases to the right of that endpoint. At the right endpoint b of the domain, the function has a local minimum since the function decreases to the left of that endpoint. The function has a local maximum at interior points of the domain c_2 and c_4 since the function increases to the left of each of these points and decreases to the right of each. The function has a local minimum at interior point of the domain c_3 since the function decreases to the left of c_3 and increases to the right of c_3. The absolute extrema are then chosen from the list of local extrema. These general ideas of "going uphill to the mountain top and then downhill" (for a local maximum) and "going downhill to the bottom of the valley and then uphill" (for a local minimum) are what is behind the First Derivative Test.

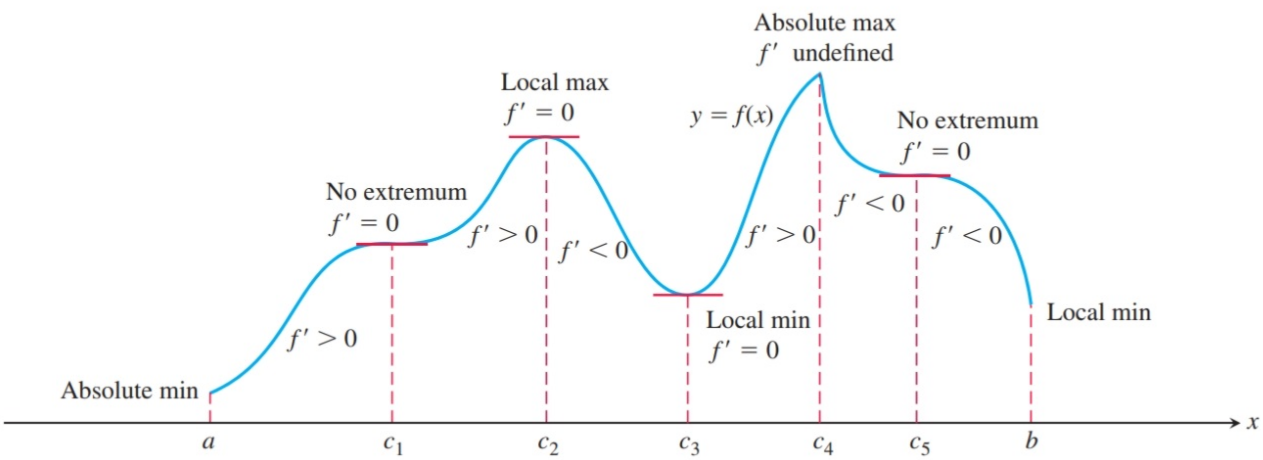

Figure 4.21

Theorem 4.3.A. First Derivative Test for Local Extrema.
Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a *local minimum* at c;

2. if f' changes from positive to negative at c, then f has a *local maximum* at c;

3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has *no local extremum* at c.

Example. Exercise 4.3.28(b).

Examples. Exercise 4.3.14, Exercise 4.3.38, Exercise 4.3.44, Exercise 4.3.58, Exercise 4.3.72, Exercise 4.3.80.

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