Chapter 5. Integrals

5.2. Sigma Notation and Limits of Finite Sums

Note. In this section we introduce a shorthand notation for summation. We will use this summation notation in the next section when we define the exact area under a curve.

Note. We use the sigma notation to denote sums:

\[ \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n. \]

The Greek letter \( \Sigma \) ("sigma," corresponding to our letter "S") stands for "sum." The index of summation \( k \) reflects where the sum begins and ends, and in general \( a_k \) is some function of \( k \) which gives the \( k \)th term of the sum:

Examples. Exercise 5.2.2 and Exercise 5.2.12.

Note. In Appendix A.2. Mathematical Induction, the following are established (see Exercise A.2.11).
Theorem 5.2.A. Algebra for Finite Sums.

1. **Sum Rule**: \( \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \)

2. **Difference Rule**: \( \sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k \)

3. **Constant Multiple Rule**: \( \sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k \)

4. **Constant Value Rule**: \( \sum_{k=1}^{n} c = nc \)

**Example.** Exercise 5.2.18.

**Note.** In Appendix A.2. Mathematical Induction, the following are established (see Example A.2.5, Exercise A.2.9, and Exercise A.2.10).

Theorem 5.2.B. The Sum of Powers of the First \( n \) Natural Numbers.

1. The first \( n \) natural numbers: \( \sum_{k=1}^{n} k = \frac{n(n + 1)}{2} \)

2. The first \( n \) natural numbers squared: \( \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6} \)

3. The first \( n \) natural numbers cubed: \( \sum_{k=1}^{n} k^3 = \left( \frac{n(n + 1)}{2} \right)^2 \).

**Examples.** Exercise 5.2.24 and Exercise 5.2.28.
Definition. A partition of the interval \([a, b]\) is a set

\[ P = \{x_0, x_1, \ldots, x_n\} \text{ where } a = x_0 < x_1 < \cdots < x_n = b. \]

Partition \(P\) determines \(n\) closed subintervals

\[[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].\]

The length of the \(k\)th subinterval is \(\Delta x_k = x_k - x_{k-1}\).

Note. We now estimate the area bounded between a function \(y = f(x)\) and the \(x\)-axis. We make the convention that the area bounded above the \(x\)-axis and below the function is positive, and the area bounded below the \(x\)-axis and above the curve is negative. We estimate this “area” by choosing a \(c_k \in [x_{k-1}, x_k]\) and we use \(f(c_k)\) as the “height” of a rectangle with base \([x_{k-1}, x_k]\). Then a partition \(P\) of \([a, b]\) can be used to estimate this “area” by adding up the “area” of these rectangles. See Figure 5.9 below.
Definition. With the above notation, a Riemann sum of \( f \) on the interval \([a, b]\) is a sum of the form

\[
s_n = \sum_{k=1}^{n} f(c_k) \Delta x_k.
\]

Example. Exercise 5.2.38.

Example 5.2.5. Partition the interval \([0, 1]\) into \( n \) subintervals of the same width, give the lower sum approximation of area under \( y = 1 - x^2 \) based on \( n \), and find the limit as \( n \to \infty \) (in which case the width of the subintervals approaches 0).
Definition. The norm of a partition $P = \{x_0, x_1, \ldots, x_n\}$ of interval $[a, b]$, denoted $\|P\|$, is length of the largest subinterval:

$$\|P\| = \max_{1 \leq k \leq n} \Delta x_k = \max_{1 \leq k \leq n} (x_k - x_{k-1}).$$

Note. If $\|P\|$ is “small,” then a Riemann sum is a “good” approximation of the “area” described above.

Figure 5.10

Note. If $[a, b]$ is partitioned into $n$ subintervals of equal length, then that length is $\Delta x_k = \Delta x = (b - a)/n$. In this case, if $n \to \infty$ then $\|P\| \to 0$.

Example. Exercise 5.2.48.