Calculus 1 Test 1 — Fall 2011

NAME KEY STUDENT NUMBER ____________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols (such as limits). Your grade is based on the work you show and the arguments you make. If applicable, put your answer in the box provided. Each numbered problem is worth 12 points. **No calculators.**

p61 Eg 3

1. (a) Find the slope of the parabola $y = x^2$ at the point $P = (2, 4)$. (b) Write an equation for the tangent to the parabola at this point. **Hint:** Choose a second point $Q = (2 + h, (2 + h)^2)$ on the curve (where $h \neq 0$) and compute the slope of the secant line PQ. *Guess* what happens to the slope of the secant line when h is close to 0.

(a) The slope of the secant line through PQ is $\frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - (4)}{(2+h) - (2)} = \frac{4 + 4h + h^2 - 4}{2 + h - 2}$ $= \frac{4h + h^2}{h} = \frac{h(4+h)}{h} = 4 + h$ since $h \neq 0$. When h is close to 0, the slope is close to 4. GUESS: $m = 4$.

(b) The tangent line is $y - y_1 = m(x - x_1)$

$y - 4 = 4(x - 2)$

$y = 4x - 4$

(a) $m = 4$
(b) $y = 4x - 4$

TYPO: $y \to -2$

p74 #18

2. Evaluate $\lim_{y \to 2} \frac{y + 2}{y^2 + 5y + 6}$, if it exists.

$$\lim_{y \to -2} \left(\frac{y+2}{y^2+5y+6}\right) \overset{F}{=} \lim_{y \to -2} \frac{y+2}{(y+2)(y+3)} \overset{C}{=} \lim_{y \to -2} \left(\frac{1}{y+3}\right) \overset{S}{=} \frac{1}{(-2)+3} = \boxed{1}$$

1

p. 77

3. State the **precise** (or **formal**) definition of $\lim_{x \to x_0} f(x) = L$ (the definition with the ϵ and δ in it).

Let $f(x)$ be defined on an open interval about x_0 except possibly at x_0 itself. We say that $f(x)$ approaches limit L as x approaches x_0 and write $\lim_{x \to x_0} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$$

p 91 #3

4. Consider $f(x) = \begin{cases} 3 - x & \text{if } x \in (-\infty, 2) \\ \frac{x}{2} + 1 & \text{if } x \in (2, \infty) \end{cases}$. Evaluate $\lim_{x \to 2} f(x)$, if it exists. Give a clear explanation of your answer!

Well, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3 - x) = 3 - (2) = 1$ and

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(\frac{x}{2} + 1\right) = \frac{(2)}{2} + 1 = 2.$$

So, $\lim_{x \to 2} f(x)$ does not exist since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$.

$\boxed{\text{Does not exist.}}$

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5. Evaluate $\lim_{x \to 0} \dfrac{x + x\cos x}{\sin x \cos x}$, if it exists.

p 92 #29

$$\lim_{x \to 0} \frac{x + x\cos(x)}{\sin(x)\cos(x)} = \lim_{x \to 0} \frac{x(1 + \cos(x))}{\sin(x)\cos(x)} = \lim_{x \to 0} \frac{x}{\sin(x)} \lim_{x \to 0} \left(\frac{1 + \cos(x)}{\cos(x)}\right)$$

$$= \lim_{x \to 0} \frac{1}{\left(\frac{\sin(x)}{x}\right)} \lim_{x \to 0} \left(\frac{1 + \cos(x)}{\cos(x)}\right) = \frac{1}{\lim_{x \to 0} \left(\frac{\sin(x)}{x}\right)} \lim_{x \to 0} \left(\frac{1 + \cos(x)}{\cos(x)}\right)$$

$$= \frac{1}{(1)} \left(\frac{1 + \cos(0)}{\cos(0)}\right) = \frac{1 + (1)}{(1)} = 2$$

$\boxed{2}$

since $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

p99
Eg10

6. Show that
$$f(x) = \frac{x^2+x-6}{x^2-4}, \quad x \neq 2$$
has a continuous extension to $x = 2$, find the extension and prove that the extension is continuous at $x = 2$.

Well, $\lim_{x\to 2}\left(\frac{x^2+x-6}{x^2-4}\right) \stackrel{F}{=} \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x+2)} \stackrel{C}{=} \lim_{x\to 2}\left(\frac{x+3}{x+2}\right) \stackrel{S}{=} \frac{(2)+3}{(2)+2} = \frac{5}{4}$

and so $f(x)$ has a removable discontinuity at $x=2$.

Define $g(x) = \frac{x+3}{x+2}$. Then $g(x) = f(x)$ for $x \neq 2$ and

$\lim_{x\to 2} g(x) = \lim_{x\to 2}\left(\frac{x+3}{x+2}\right) = \frac{(2)+3}{(2)+2} = \frac{5}{4} = g(2)$, so g is

continuous at $x=2$ and hence g is the continuous extension of f.

$$\boxed{g(x) = \frac{x+3}{x+2}}$$

p115
#36

7. Evaluate $\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}}$.

$\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to-\infty}\left(\frac{4-3x^3}{\sqrt{x^6+9}}\right)\left(\frac{1/x^3}{1/x^3}\right) = \lim_{x\to-\infty} \frac{\left(\frac{4}{x^3} - \frac{3x^3}{x^3}\right)}{\left(\frac{\sqrt{x^6+9}}{x^3}\right)}$

$= \lim_{x\to-\infty} \frac{\left(\frac{4}{x^3} - \frac{3x^3}{x^3}\right)}{\sqrt{\frac{x^6+9}{x^6}}} = \lim_{x\to-\infty}\left(\frac{\frac{4}{x^3} - 3}{\sqrt{1+\frac{9}{x^6}}}\right) = \frac{4\left(\lim_{x\to-\infty}\frac{1}{x}\right)^3 - 3}{\sqrt{1+9\left(\lim_{x\to-\infty}\frac{1}{x}\right)^6}}$

$= \frac{4(0)^3-3}{\sqrt{1+9(0)^6}} = \frac{-3}{1} = -3$

$$\boxed{-3}$$

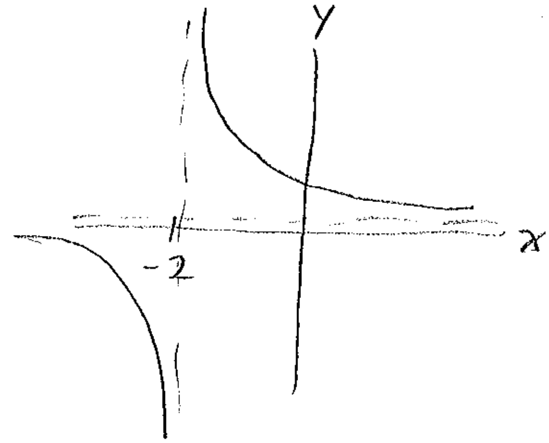
p115 #65

8. Consider $f(x) = \dfrac{1}{2x+4}$. Then $\lim_{x\to\pm\infty} f(x) = 0$. Use limits to find the vertical asymptotes of f and graph $y = f(x)$ in such a way as to reflect the asymptotes.

We check $f(x)$ for V. A. at $x=-2$ (where the denominator is 0):

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-}\left(\frac{1}{2x+4}\right) \overset{\frac{(+)}{(-)}}{=\!=} -\infty$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+}\left(\frac{1}{2x+4}\right) \overset{\frac{(+)}{(+)}}{=\!=} +\infty.$$



So $x=-2$ is a V. A. of f. Also, since $\lim_{x\to\pm\infty} f(x) = 0$, then $y=0$ is a H. A. of f.

$$\boxed{\begin{array}{l}\text{VA at } x=-2.\\ \text{HA of } y=0.\end{array}}$$

p102 #57b

Bonus. If $f(x) = x^3 - 8x + 10$, show that there is a value c for which $f(c) = -\sqrt{3}$. Explain your answer.

Well, $f(x)$ is a polynomial and so is continuous everywhere.

Next, $f(0) = 10$ and $f(-4) = (-4)^3 - 8(-4) + 10 = -64 + 32 + 10 = -22$.

Since $f(0) = 10 > -\sqrt{3}$ and $f(-4) = -22 < -\sqrt{3}$, then by the Intermediate Value Theorem, there exists $c \in [-4, 0]$ such that $f(c) = -\sqrt{3}$.