Calculus 1 Test 1 — Summer 2013

NAME KEY STUDENT NUMBER ____________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols (such as limits). If applicable, put your answer in the box provided. Each numbered problem is worth 10 points. You may use your calculator any way you see fit, but **you may not share calculators**!!! Please put away your cell phones.

1. Find the average rate of change of $h(t) = \cot t$ on the interval $[\pi/6, \pi/2]$. Evaluate the trig functions and give exact values.

p 63 #3

Well,

$$\begin{pmatrix}\text{average}\\ \text{rate of}\\ \text{change}\end{pmatrix} = \frac{h(t_2)-h(t_1)}{t_2-t_1} = \frac{\cot\left(\frac{\pi}{2}\right)-\cot\left(\frac{\pi}{6}\right)}{\frac{\pi}{2}-\frac{\pi}{6}} = \frac{0-\sqrt{3}}{\frac{\pi}{3}} = -\frac{3\sqrt{3}}{\pi}$$

$$\boxed{-\frac{3\sqrt{3}}{\pi}}$$

2. Evaluate $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$.

p 74 #37

$$\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x+3}-2}\right)\left(\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right) = \lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4}$$

$$\stackrel{F}{=} \lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(x-1)} \stackrel{C}{=} \lim_{x\to 1}\left(\sqrt{x+3}+2\right) \stackrel{S}{=} \sqrt{(1)+3}+2 = 4$$

$$\boxed{4}$$

p.77

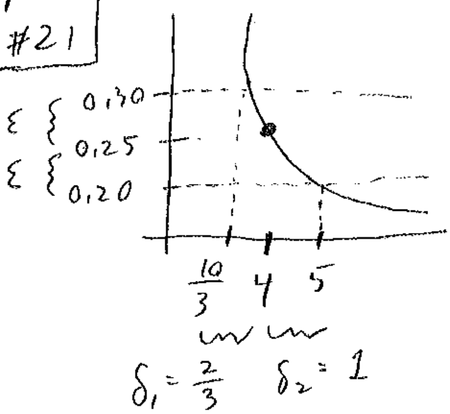
3. State the **precise** (or **formal**) definition of $\lim_{x\to x_0} f(x) = L$ (the definition with the ϵ and δ in it).

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. Then $\lim_{x\to x_0} f(x) = L$ if, for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

p83 #21

4. Suppose $f(x) = 1/x$, $x_0 = 4$, and $L = 1/4$. For $\epsilon = 0.05$, find a corresponding $\delta > 0$ which is guaranteed to exist by the formal definition of limit.



Well, $f(x) = \frac{1}{x} = L + \varepsilon = 0.30 \Rightarrow x = \frac{10}{3}$

and $f(x) = \frac{1}{x} = L - \varepsilon = 0.20 \Rightarrow x = 5$.

So δ is the smaller of $\delta_1 = 4 - \frac{10}{3} = \frac{2}{3}$ and $\delta_2 = 5 - 4 = 1$.

Hence, $\delta = \frac{2}{3}$ (or anything smaller).

$$\boxed{\delta = \frac{2}{3}}$$

p86 Eg. 1

5. Consider $f(x) = \sqrt{4 - x^2}$. Evaluate $\lim_{x\to 2} f(x)$. Give a clear explanation of your answer and heed any warning signs!

Well, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \sqrt{4 - x^2} = \lim_{x\to 2^+} \sqrt{\underset{+}{(2-x)}\underset{+}{(2+x)}} = \sqrt{0} = 0$

WARNING SIGN! HEEDED

Also, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \sqrt{4 - x^2} = \lim_{x\to 2^+} \sqrt{\underset{-}{(2-x)}\underset{+}{(2+x)}}$ does not exist (square roots of negatives).

So $\lim_{x\to 2} f(x)$ does not exist since

$\lim_{x\to 2^+} f(x)$ does not exist

$$\boxed{\text{does not exist}}$$

6. Evaluate $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin 2\theta}$.

p92 #31

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\sin(2\theta)} = \lim_{\theta \to 0} \left(\frac{(2\theta)}{\sin(2\theta)}\right)\left(\frac{1-\cos\theta}{2\theta}\right)$$

$$= \lim_{\theta \to 0} \frac{2\theta}{\sin(2\theta)} \lim \left(\frac{1}{2} \frac{1-\cos\theta}{\theta}\right) = \frac{1}{2} \lim_{\theta \to 0} \frac{2\theta}{\sin(2\theta)} \cdot \lim_{\theta \to 0}\left(\frac{1-\cos\theta}{\theta}\right)$$

$$= \frac{1}{2}(1)(0) = 0$$

$$\boxed{0}$$

7. Consider $f(x) = \begin{cases} x^2 - 1 & \text{if } x \in [-1, 0) \\ 2x & \text{if } x \in (0,1) \\ 1 & \text{if } x = 1 \\ -2x + 4 & \text{if } x \in (1,2) \\ 0 & \text{if } x \in (2,3) \end{cases}$. Is f continuous at $x = 2$? Use the Test for Continuity and give a detailed explanation. If it is not continuous, what type of discontinuity does it have?

p101 #7

Well, $f(2)$ is not defined, so NO f is not continuous at $x=2$.

Now $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (-2x+4) = -2(2)+4 = 0$

and $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (0) = 0$, so $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 0$

and $\lim_{x \to 2} f(x) = 0$. So f has a removable discontinuity at $x=2$.

$$\boxed{\text{NOT continuous; removable discontinuity}}$$

8. Consider $y = \dfrac{x+3}{x+2}$. Find the horizontal asymptote and all vertical asymptote(s). Graph $y = f(x)$ in such a way as to reflect the asymptotic behavior.

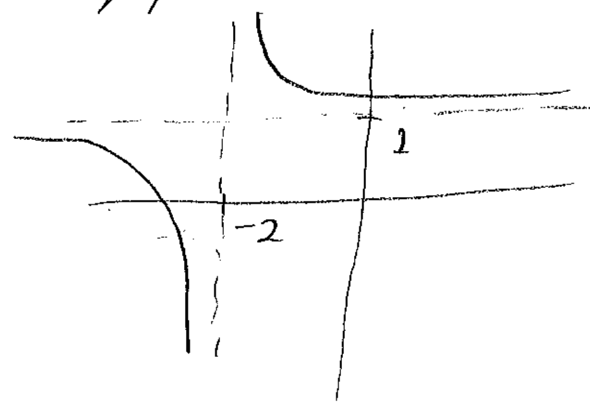
p115 #67

For H.A., $\displaystyle\lim_{x\to\pm\infty}\left(\frac{x+3}{x+2}\right) = \lim_{x\to\pm\infty}\left(\frac{x+3}{x+2}\right)\left(\frac{1/x}{1/x}\right) = \lim_{x\to\pm\infty}\left(\frac{1+\frac{3}{x}}{1+\frac{2}{x}}\right)$

$$= \frac{1 + 3\left(\lim_{x\to\pm\infty}\frac{1}{x}\right)}{1 + 2\left(\lim_{x\to\pm\infty}\frac{1}{x}\right)} = \frac{1+3(0)}{1+2(0)} = 1.$$

So $y = 1$ is H.A.

The graph is



For V.A., consider

$$\lim_{x\to -2^-}\left(\frac{x+3}{x+2}\right) = -\infty$$

$$\lim_{x\to -2^+}\left(\frac{x+3}{x+2}\right) = +\infty$$

So $x = -2$ is a V.A.

$$\boxed{\begin{array}{l} \text{H.A. } y = 1 \\ \text{V.A. } x = -2 \end{array}}$$

9. Find the slope of the tangent to the curve $y = (x-1)^2 + 1$ at the point $(1,1)$. **Use the definition of slope of a tangent line** to a curve!

p125 #6

$$m = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{h\to 0}\frac{\left((1+h)-1)^2+1\right) - \left(((1)-1)^2+1\right)}{h}$$

$$= \lim_{h\to 0}\left(\frac{h^2+1-1}{h}\right) \overset{F}{=} \lim_{h\to 0}\left(\frac{h^2}{h}\right) \overset{C}{=} \lim_{h\to 0} h \overset{S}{=} 0.$$

$$\boxed{0}$$

10. Use the definition of derivative (which involves limits) to differentiate $k(x) = \frac{1}{2+x}$.

p132 #14

$$k'(x) = \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{2+(x+h)} - \frac{1}{2+x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{(2+x)-(2+x+h)}{(2+x+h)(2+x)}\right) \overset{F}{=} \lim_{h\to 0} \frac{1}{h} \frac{-h}{(2+x+h)(2+x)}$$

$$\overset{C}{=} \lim_{h\to 0} \frac{-1}{(2+x+h)(2+x)} \overset{S}{=} \frac{-1}{(2+x+(0))(2+x)} = \frac{-1}{(2+x)^2}$$

$$\boxed{\frac{-1}{(2+x)^2}}$$

Bonus. (a) State the Intermediate Value Theorem. (b) Explain why the equation $x^3 - 8x + 10 = -\sqrt{3}$ has a solution.

p102 #57b

(a) A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. That is, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

(b) Define $f(x) = x^3 - 8x + 10$. Then $f(x)$ is continuous since it is a polynomial. Now $f(0) = 10 > -\sqrt{3}$ and $f(-4) = (-4)^3 - 8(-4) + 10 = -22 < -\sqrt{3}$. So, by the Intermediate Value Theorem, there exists $c \in [-4, 0]$ such that $f(c) = -\sqrt{3}$.