Calculus 1 Test 2 (Online) — Fall 2020

NAME KEY STUDENT NUMBER ____________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols (such as limits). **Write in complete sentences!** When asked to explain, give justifications using the definitions and theorems in the notes and book (quote them by name or number, as we did in class). Your grade is based on the work you show and the arguments you make. If applicable, put your final answer in the box provided. Each numbered problem is worth 20 points.

You are not to collaborate with anyone during the test! When you are done, scan your solutions and put them in the Dropbox in D2L. If you have trouble with D2L (which I wouldn't expect), then e-mail me your solutions at `gardnerr@etsu.edu` (but please try to get your solutions submitted in D2L). I prefer PDFs, but if things aren't working then go ahead and just submit photographs of your solutions. The test is due by 11:15.

1. Consider the function $f(x) = \begin{cases} 2x, & x \in (-\infty, 1) \\ 0, & x = 1 \\ 3x-1 & x \in (1,\infty). \end{cases}$ Use the Test for Continuity to see if f is continuous at $x = 1$. Show all details.

The Test for Continuity requires ① $f(1)$ to exist, ② $\lim_{x\to 1} f(x)$ to exist, and ③ $\lim_{x\to 1} f(x) = f(1)$. First, $f(1) = 0$ exists. Second, since f is defined piecewise we consider one-sided limits:

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x) = 2(1) = 2$ since $2x$ is a polynomial (Theorem 2.2)

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3x-1) = 3(1)-1 = 2$ since $3x-1$ is a polynomial.

So $\lim_{x\to 1} f(x) = 2$ by Theorem 2.6 (Relation Between 1-Sided and 2-Sided Limits) and ② holds. BUT $\lim_{x\to 1} f(x) = 2 \neq 0 = f(1)$ so ③ is violated and f is NOT continuous at $x=1$.

$\boxed{\text{NO, } \lim_{x\to 1} f(x) \neq f(1)}$

2. Find all horizontal and vertical asymptotes of $f(x) = \dfrac{x+5}{x+1}$. Use the definitions of horizontal and vertical asymptote, and find all relevant limits.

For horizontal asymptotes, $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(\frac{x+5}{x+1}\right) = \lim_{x\to\pm\infty}\left(\frac{x+5}{x+1}\right)\left(\frac{1/x}{1/x}\right)$

$$= \lim_{x\to\pm\infty}\left(\frac{x/x+5/x}{x/x+1/x}\right) = \lim_{x\to\pm\infty}\left(\frac{1+5/x}{1+1/x}\right) = \frac{1+5\left(\lim_{x\to\pm\infty} 1/x\right)}{1+\left(\lim_{x\to\pm\infty} 1/x\right)}$$

by the Sum Rule, Quotient Rule, and Constant Multiple Rule of Theorem 2.12

$$= \frac{1+5(0)}{1+(0)} = 1 \text{ since } \lim_{x\to\pm\infty}(1/x) = 0 \text{ by Example 2.6.1,}$$

so $y=1$ is a horizontal asymptote. For vertical asymptotes, we consider (for the rational function f) where the denominator is 0 ; $x+1=0$ or $x=-1$. Since the numerator $x+5$ is not 0 when $x=-1$, then by Dr. Bob's Infinite Limits Theorem we have $\lim_{x\to -1^-} f(x) = \pm\infty$ and $\lim_{x\to -1^+} f(x) = \pm\infty$. So $x=-1$ is a vertical asymptote. We still need to know the values of these two one-sided limits.

Finally, by "sign diagrams"

for $x\to -1^-$ we have $\dfrac{x+5}{x+1} \Rightarrow \dfrac{(+)}{(-)} = -$

and for $x\to -1^+$ we have $\dfrac{x+5}{x+1} \Rightarrow \dfrac{(+)}{(+)} = +$.

So $\lim_{x\to -1^-} f(x) = -\infty$ and $\lim_{x\to -1^+} f(x) = +\infty$.

H. A. of $y=1$.
V. A. of $x=-1$ where $\lim_{x\to -1^-} f(x) = -\infty$ and $\lim_{x\to -1^+} f(x) = \infty$

3. Use the limit definition of derivative to differentiate $f(x) = \sqrt{x^2+1}$.

By definition, we consider the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} \right) \left(\frac{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \right)$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{(x+h)^2+1}\right)^2 - \left(\sqrt{x^2+1}\right)^2}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)} = \lim_{h \to 0} \frac{((x+h)^2+1) - (x^2+1)}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h \to 0} \frac{x^2+2xh+h^2+1-x^2-1}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)} = \lim_{h \to 0} \frac{2xh+h^2}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$\overset{F}{=} \lim_{h \to 0} \frac{h(2x+h)}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)} \overset{C}{=} \lim_{h \to 0} \frac{2x+h}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}$$

$$\overset{S}{=} \frac{2x+(0)}{\sqrt{(x+(0))^2+1} + \sqrt{x^2+1}}$$ by the Sum Rule, Quotient Rule, Root Rule, and Power Rule of Theorem 2.1

$$= \frac{2x}{\sqrt{x^2+1} + \sqrt{x^2+1}} = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}.$$

$$\boxed{f'(x) = \frac{x}{\sqrt{x^2+1}}.}$$

4. Using the rules of differentiation and my square bracket notation (but you do not need to simplify), find the derivatives of **(a)** $f(x) = \dfrac{1}{x^{1.4}} + \dfrac{\pi}{\sqrt{x}}$, and **(b)** $g(x) = \dfrac{(x^2+x)(5x^2-x+4)}{3x^3-2}$. Put a box around your answer.

(a) We have $f(x) = x^{-1.4} + \pi x^{-1/2}$, so by the Sum Rule and Power Rule (Theorems 3.3.E and 3.3.C, respectively)

$$f'(x) = -1.4x^{-2.4} + \pi\left[-\tfrac{1}{2}x^{-3/2}\right] = \boxed{\frac{-1.4}{x^{2.4}} - \frac{\pi}{2x^{3/2}}}.$$

(b) By the Quotient Rule (Theorem 3.3.H) and the Product Rule (Theorem 3.3.G)

$$\boxed{g'(x) = \Big\{\Big[[2x+1](5x^2-x+4) + (x^2+x)[10x-1]\Big](3x^3-2) - \big((x^2+x)(5x^2-x+4)\big)[9x^2]\Big\} \Big/ (3x^3-2)^2.}$$

5. **(a)** Does the graph of $y = x - \tan x$ have a horizontal tangent in $(-\pi/2, \pi/2)$? Explain your answer.

The derivative $y' = 1 - \sec^2(x)$ gives the slope of the line tangent to y and a horizontal line has slope 0, so we set $y' = 1 - \sec^2(x) \equiv 0$. This implies $\sec^2(x) = 1$ or $\sqrt{\sec^2(x)} = \sqrt{1}$ or $|\sec(x)| = 1$ or $\sec(x) = \pm 1$ or $\frac{1}{\cos(x)} = \pm 1$ or $\cos(x) = \pm 1$. This has a solution in $(-\pi/2, \pi/2)$ of $x = 0$ since $\cos(0) = 1$. So $y = x - \tan(x)$ has a tangent line at $x = 0$.

(a) $\boxed{\text{YES, at } x = 0 \text{ (and } y = 0\text{).}}$

(b) Let $y = \sin(x^2 e^{3x})$. Use the rules of differentiation and my square bracket notation to find y' and y''. Put a box around your answer.

By the Chain Rule (Theorem 3.2) and the Product Rule (Thm 3.3.6

$$y' = \cos(x^2 e^{3x}) \left[[2x](e^{3x}) + (x^2)[e^{3x}[3]] \right]$$

$$= \boxed{2xe^{3x}\cos(x^2e^{3x}) + 3x^2e^{3x}\cos(x^2e^{3x}) = y'.}$$

By the Product Rule for a product of 3 functions (see Exercise 3.3.77(a)):

$$\boxed{\begin{aligned} y'' = {} & [2](e^{3x})(\cos(x^2e^{3x})) + (2x)[e^{3x}[3]](\cos(x^2e^{3x})) \\ & + (2x)(e^{3x})\left[-\sin(x^2e^{3x})\left[[2x](e^{3x}) + (x^2)[e^{3x}[3]]\right]\right] \\ & + [6x](e^{3x})(\cos(x^2e^{3x})) + (3x^2)[e^{3x}[3]](\cos(x^2e^{3x})) \\ & + (3x^2)(e^{3x})\left[-\sin(x^2e^{3x})\left[[2x](e^{3x}) + (x^2)[e^{3x}[3]]\right]\right] \end{aligned}}$$