Honors Calculus 1 Test 3 — Fall 2011

NAME KEY STUDENT NUMBER ______________________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols (such as limits). Your grade is based on the work you show and the arguments you make. If applicable, put your answer in the box provided. Each numbered problem is worth 12 points. **No calculators.**

1. (a) Differentiate: $y = t\tan^{-1}(t) - \frac{1}{2}\ln t$.

$\boxed{\text{p214 \#59}}$

$$y' = [1](\tan^{-1}(t)) + (t)\left[\frac{1}{1+t^2}\right] - \frac{1}{2}\left[\frac{1}{t}\right]$$

$$\boxed{\tan^{-1}(t) + \frac{t}{1+t^2} - \frac{1}{2t}}$$

$\boxed{\text{p203 Eg. 3}}$

(b) Find the linearization of $f(x) = \cos x$ at $a = \pi/2$.

Well, $L(x) = f'(a)(x-a) + f(a)$ and

$f(a) = \cos\left(\frac{\pi}{2}\right) = 0$; $f'(x) = -\sin x \Rightarrow f'(a) = -\sin\left(\frac{\pi}{2}\right) = -1$.

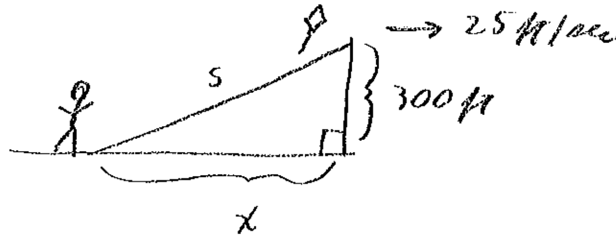
So, $L(x) = -1\left(x - \frac{\pi}{2}\right) + 0 = -x + \frac{\pi}{2}$

$$\boxed{L(x) = -x + \frac{\pi}{2}}$$

p199 #25

2, 3. A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her. Use the six step method described in the class notes!

(1)



(2) We know $\frac{dx}{dt} = 25$ ft/sec

(3) The question is $\frac{ds}{dt} = ?$ when $s = 500$ ft.

(4) By the Pythagorean Theorem $x^2 + (300\text{ ft})^2 = s^2$.

(5) Differentiate implicitly w.r.t. time:

$$\frac{d}{dt}\left[x^2 + (300)^2\right] = \frac{d}{dt}\left[s^2\right] \Rightarrow 2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$

$$\Rightarrow \frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$$

(6) WHEN $s = 500$ ft, $x^2 + (300\text{ ft})^2 = (500\text{ ft})^2 \Rightarrow x = 400$ ft

and $\frac{ds}{dt} = \frac{(400\text{ ft})}{(500\text{ ft})}(25\text{ ft/sec}) = 20$ ft/sec

$\boxed{20 \text{ ft/sec}}$

4. Find the absolute maximum and minimum values of $f(x) = -3x^{2/3}$ on the interval $[-1, 1]$. Find the critical points of f and show all appropriate details.

$\boxed{p\,228 \quad \#28}$

Well, $f'(x) = -3\left[\frac{2}{3}x^{-1/3}\right] = \dfrac{-2}{x^{1/3}} \Rightarrow x = 0$ is a critical point.

So, consider

x	$f(x)$
-1	$-3(-1)^{2/3} = -3$
0	$-3(0)^{2/3} = 0$
1	$-3(1)^{2/3} = -3$

So f has a MAX of 0 at $x = 0$

f has a MIN of -3 at $x = -1$ and $x = 1$.

$\boxed{\text{MAX } 0 \text{ at } x = 0 \quad \text{MIN } -3 \text{ at } x = -1, +1}$

5. Show that $f(x) = x^{4/5}$ satisfies the hypotheses of the Mean Value Theorem on $[0, 1]$. Find the number $c \in [0, 1]$ which is guaranteed to exist by the Mean Value Theorem.

$\boxed{p\,236 \quad \#10}$

Well, $f(x) = x^{4/5}$ is continuous on $[0,1]$ (in fact, is continuous on $\mathbb{R} = (-\infty, \infty)$). $f'(x) = \frac{4}{5}x^{-1/5} = \dfrac{4}{5x^{1/5}}$,

so f is differentiable on $(0,1)$ (in fact, f is differentiable on $(-\infty, 0) \cup (0, \infty)$). So, set

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow \frac{4}{5c^{1/5}} = \frac{(1)^{4/5} - (0)^{4/5}}{(1) - (0)} = 1$$

$$\Rightarrow c^{1/5} = \frac{4}{5} \Rightarrow c = \left(\frac{4}{5}\right)^5 \in (0,1).$$

$\boxed{c = \left(\frac{4}{5}\right)^5}$

6. Find the function g with derivative $g'(x) = \frac{1}{x^2} + 2x$ and with a graph which passes through the point $P(-1, 1)$.

p237 #40

Well, $g'(x) = x^{-2} + 2x \Rightarrow g(x) = -x^{-1} + x^2 + h$ for some constant h.

Now $g(-1) = 1 \Rightarrow -(-1)^{-1} + (-1)^2 + h = 1 = 2 + h = 1$

$\Rightarrow h = -1$. So, $g(x) = -x^{-1} + x^2 - 1$

$$\boxed{g(x) = \frac{-1}{x} + x^2 - 1}$$

7, 8. Consider $f(x) = -2x^3 + 6x^2 - 3$. Find: The critical points of f, the intervals on which f is INC/DEC, the extrema of f, the intervals on which f is CU/CD, the points of inflection of f, and graph $y = f(x)$.

p252 #13

Well, $f'(x) = -6x^2 + 12x = 6x(2-x)$, so $\boxed{x = 0,\ x = 2 \text{ are critical points.}}$

Consider :

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
h	-1	1	3
$f'(h)$	-18	6	-18
$f'(x)$	$-$	$+$	$-$
f	DEC	INC	DEC

So, $\boxed{\begin{array}{l} f \text{ is INC on } (0,2) \\ f \text{ is DEC on } (-\infty, 0) \cup (2, \infty) \end{array}}$

$\Rightarrow$ $\boxed{\begin{array}{l} f \text{ has a MAX at } x = 2 \text{ of } f(2) = 5 \\ f \text{ has a MIN at } x = 0 \text{ of } f(0) = -3 \end{array}}$

Next, $f''(x) = -12x + 12 = 12(1-x) \Rightarrow$

$x = 1$ is a potential point of inflection.

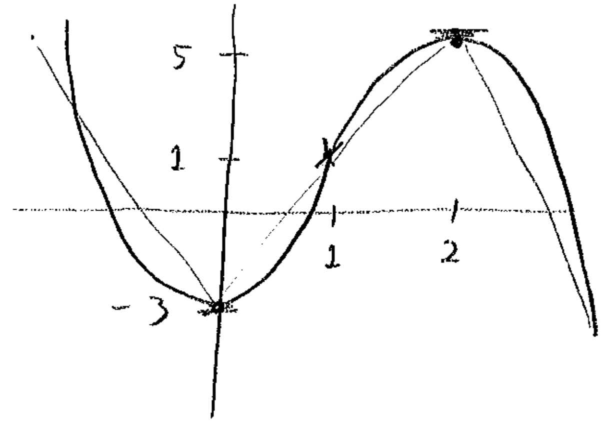
Consider

	$(-\infty, 1)$	$(1, \infty)$
h	0	2
$f''(h)$	12	-12
$f''(x)$	$+$	$-$
f	CU	CD

$\boxed{\begin{array}{l} \text{So, } f \text{ is CU on } (-\infty, 1) \\ f \text{ is CD on } (1, \infty) \end{array}}$

$\boxed{f \text{ has a point of inflection at } x = 1}$

Notice $f(1) = 1$.



Honors Bonus. Use the fact that two functions with the same derivative differ by a constant to show that $\ln ax = \ln a + \ln x$ for all positive a and x.

$\boxed{p234}$

Notice that

$$\frac{d}{dx}\left[\ln(ax)\right] = \frac{1}{ax}[a] = \frac{1}{x}$$

and $$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}.$$

So, $\ln(ax) = \ln(x) + k$ for some constant k.

With $x = 1$, we see that

$$\ln(a \cdot 1) = \ln(1) + k \Rightarrow \ln(a) = k.$$

So, $\ln(ax) = \ln(x) + \ln(a)$. $\blacksquare$