Calculus 1 Test 3 — Fall 2012

NAME KEY STUDENT NUMBER

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols (such as limits). Your grade is based on the work you show and the arguments you make. If applicable, put your answer in the box provided. Each numbered problem is worth 12 points. **No calculators.**

p210 #5

1. (a) Find the linearization of $f(x) = \tan x$ at $a = \pi$.

Well, $f'(x) = \sec^2(x)$ and so $f'(a) = \sec^2(\pi) = \frac{1}{\cos^2(\pi)} = \frac{1}{(-1)^2} = 1$.

So, $L(x) = f(a) + f'(a)(x-a) = \tan(\pi) + (1)(x-\pi)$

$= 0 + (x-\pi) = (x-\pi)$

$\boxed{L(x) = x - \pi}$

p261 #45

(b) Use L'Hôpital's Rule to evaluate $\lim_{t\to\infty} \frac{e^t + t^2}{e^t - t}$. Show that you have checked the hypotheses of L'Hôpital's Rule.

$$\lim_{t\to\infty}\left(\frac{e^t+t^2}{e^t-t}\right) \overset{\infty/\infty}{=} \lim_{t\to\infty}\left(\frac{e^t+2t}{e^t-1}\right) \overset{\infty/\infty}{=} \lim_{t\to\infty}\left(\frac{e^t+2}{e^t}\right)$$

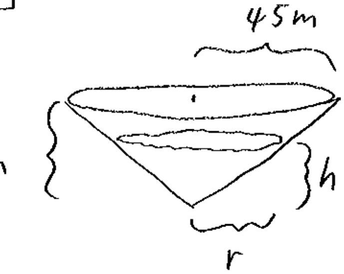
$$\overset{\infty/\infty}{=} \lim_{t\to\infty}\left(\frac{e^t}{e^t}\right) = \lim_{t\to\infty}(1) = 1$$

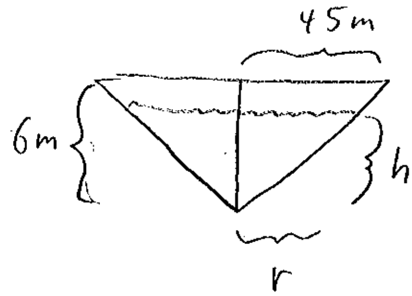
$\boxed{1}$

$\boxed{\text{p 199 \#28}}$

2, 3. Water is flowing at the rate of 50 m^3/min from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m. How fast is the water level falling when the water is 5 m deep? HINT: The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

(1)





(2,3) The question is $\frac{dh}{dt} = ?$ when $h = 5\text{ m}$ and $\frac{dV}{dt} = -50\ m^3/min$.

(4) By similar triangles, we know $\frac{h}{r} = \frac{6}{45}$ or $r = \frac{45}{6}h = \frac{15}{2}h$.

So $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{15}{2}h\right)^2 h = \frac{75}{4}\pi h^3$.

(5) Differentiate WRT t: $\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{75}{4}\pi h^3\right]$

or $\frac{dV}{dt} = \frac{75}{4}\pi\, 3h^2\left[\frac{dh}{dt}\right] = \frac{225\pi}{4} h^2 \frac{dh}{dt}$

or $\frac{dh}{dt} = \frac{4}{225\pi h^2}\frac{dV}{dt}$,

(6) When $h = 5\text{ m}$ and $\frac{dV}{dt} = -50\ m^3/min$,

$$\frac{dh}{dt} = \frac{4}{225\pi(5m)^2}\left(-50\ m^3/min\right) = \frac{-200}{225\pi(25)}\ m/min = \frac{-8}{225\pi}\ m/min$$

$$\boxed{\frac{-8}{225\pi}\ m/min}$$

p 228 #29

4. Find the absolute maximum and minimum values of $g(x) = \sqrt{4-x^2}$ on the interval $[-2, 1]$. Find the critical points of g and show all appropriate details.

Well, $g'(x) = \frac{1}{2}(4-x^2)^{-1/2}[-2x] = \frac{-x}{\sqrt{4-x^2}}$.

Set $g'(x) = \frac{-x}{\sqrt{4-x^2}} \equiv 0$ and $x = 0$ is a critical point. Also, $x=-2$ and $x=2$ are critical points.

Consider

x	$g(x)$
-2	0
0	2
1	$\sqrt{3}$

So g has a MAX at $x=0$ of 2
g has a MIN at $x=-2$ of 0.

$$\boxed{\begin{aligned} MAX &= 2 \\ MIN &= 0 \end{aligned}}$$

5. Show that $f(x) = x^{4/5}$ satisfies the hypotheses of the Mean Value Theorem on $[0, 1]$. Find the number $c \in [0, 1]$ which is guaranteed to exist by the Mean Value Theorem.

p 236 #10

Well, $f(x) = x^{4/5}$ is continuous on its domain $\mathbb{R}$, so f is continuous on $[0,1]$. Next, $f'(x) = \frac{4}{5}x^{-1/5} = \frac{4}{5\sqrt[5]{x}}$, so f is differentiable except at $x=0$ — that is, f is differentiable on $(0,1)$. So the hypotheses of MVT are satisfied. Next, set

$$f'(c) = \frac{f(b)-f(a)}{b-a}, \text{ or } \frac{4}{5\sqrt[5]{c}} = \frac{(1)^{4/5}-(0)^{4/5}}{(1)-(0)}$$

or $\frac{4}{5\sqrt[5]{c}} = 1 \Rightarrow \sqrt[5]{c} = \frac{4}{5}$ and

$$\boxed{c = \left(\frac{4}{5}\right)^5}$$

$c = \left(\frac{4}{5}\right)^5 \in (0,1)$

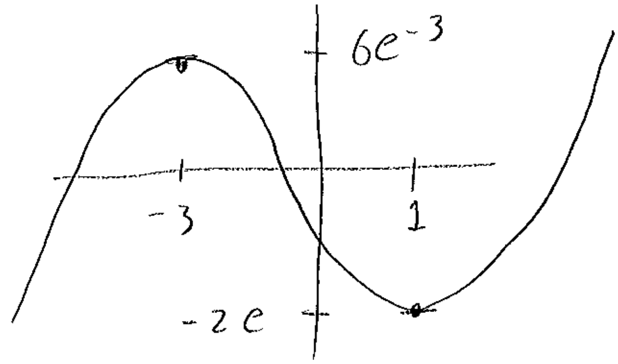
p 241, Eg 3

6. Consider $f(x) = (x^2-3)e^x$. Find: The critical points of f, the intervals on which f is INC/DEC, the extrema of f, and graph $y = f(x)$.

Well, $f'(x) = [2x](e^x) + (x^2-3)[e^x] = e^x(2x + x^2 - 3)$
$= e^x(x^2+2x-3) = e^x(x+3)(x-1)$. So $x = 1$ and $x = -3$ are critical points.

Consider:

	$(-\infty,-3)$	$(-3,1)$	$(1,\infty)$
h	-4	0	2
$f'(h)$	$(+)(-)(-)$	$(+)(+)(-)$	$(+)(+)(+)$
$f'(x)$	$+$	$-$	$+$
$f(x)$	INC	DEC	INC



So f has a MAX at $x = -3$ of $f(-3) = 6e^{-3}$
f has a MIN at $x = 1$ of $f(1) = -2e$

p 252 #19

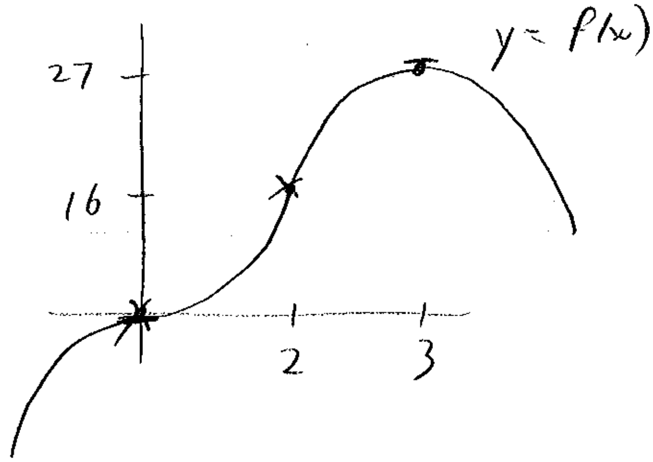
7. Consider $f(x) = 4x^3 - x^4 = x^3(4-x)$. Find: The critical points of f, the second derivative, the intervals on which f is CU/CD, the points of inflection, and graph $y = f(x)$.

Well, $f'(x) = 12x^2 - 4x^3 = 4x^2(3-x) \Rightarrow$ $x = 0, 3$ are critical points

Next, $f''(x) = 24x - 12x^2 = 12x(2-x) \Rightarrow x = 0, 2$ are potential points of inflection.

Consider

	$(-\infty,0)$	$(0,2)$	$(2,\infty)$
h	-1	1	3
$f''(h)$	$(+)(+)$	$(+)(+)$	$(+)(-)$
$f''(x)$	$-$	$+$	$-$
$f(x)$	CD	CU	CD



Notice $f(0) = 0$, $f(2) = 16$, $f(3) = 27$

So $x = 0$ and $x = 2$ are points of inflection.

p261 #59

8. Use L'Hôpital's Rule to evaluate $\lim_{x\to 0+} x^x$. Show that you have checked the hypotheses of L'Hôpital's Rule.

The limit is of the 0^0 indeterminate form. So let $y = x^x$.
Then $\ln(y) = \ln(x^x) = x\ln(x)$. So

$$\lim_{x\to 0^+}(\ln y) = \lim_{x\to 0^+}(x\ln x) = \lim_{x\to 0^+}\left(\frac{\ln x}{1/x}\right) \overset{\infty/\infty}{=} \lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)$$

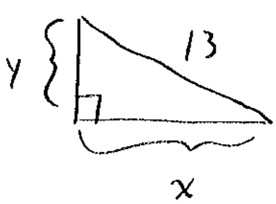
$$= \lim_{x\to 0^+}(-x) = 0.$$ So $$e^{\lim_{x\to 0^+}(\ln y)} = e^0 = 1$$ or

$$\lim_{x\to 0^+}\left(e^{\ln y}\right) = 1$$ or $$\lim_{x\to 0^+}(y) = \lim_{x\to 0^+}(x^x) = 1$$

$$\boxed{1}$$

Bonus. A 13 ft ladder is leaning against a house when it base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec. How fast is the top of the ladder sliding down the wall then?

p198 #23a

(1) 

$\rightarrow \frac{dx}{dt} = 5$ ft/sec

(2) We know $\frac{dx}{dt} = 5$ ft/sec when $x = 12$ ft

(3) The question is $\frac{dy}{dt} = ?$ when $x = 12$ ft and $\frac{dx}{dt} = 5$ ft/sec.

(4) We know $x^2 + y^2 = 13^2 = 169$.

(5) Differentiate WRT t: $\frac{d}{dt}[x^2+y^2] = \frac{d}{dt}[169]$

or $2x\left[\frac{dx}{dt}\right] + 2y\left[\frac{dy}{dt}\right] = 0$, or

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

$$\boxed{-12 \text{ ft/sec}}$$

(6) When $x = 12$ ft, then $y = 5$ ft and $\frac{dy}{dt} = \frac{-(12\text{ ft})}{(5\text{ ft})}(5\text{ ft/sec}) = -12$ ft/sec