Calculus 1 Test 3 (Online) — Fall 2020

NAME KEY STUDENT NUMBER ____________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols. **Write in complete sentences!** Your grade is based on the work you show and the arguments you make. If applicable, put your final answer in the box provided. Each numbered problem is worth 20 points.

You are not to collaborate with anyone during the test! When you are done, scan your solutions and put them in the Dropbox in D2L. If you have trouble with D2L (which I wouldn't expect), then e-mail me your solutions at `gardnerr@etsu.edu` (but please try to get your solutions submitted in D2L). I prefer PDFs, but if things aren't working then go ahead and just submit photographs of your solutions. The test is due by 11:15 but you can still submit to DropBox until 11:30.

1. Consider the equation $x^4 \cos y - \sin^2 y = 0$. Find dy/dx.

We differentiate implicitly with respect to x to get

$$\frac{d}{dx}\left[x^4 \cos(y) - \sin^2(y)\right] = \frac{d}{dx}[0] \text{ or}$$

$$[4x^3](\cos(y)) + (x^4)\left[-\sin(y)\left[\frac{dy}{dx}\right]\right] - 2\sin(y)\left[\cos(y)\left[\frac{dy}{dx}\right]\right] = 0$$

or $$\frac{dy}{dx}\left(-x^4 \sin y - 2\cos y \sin y\right) = -4x^3 \cos(y)$$

or $$\frac{dy}{dx} = \frac{-4x^3 \cos y}{-x^4 \sin y - 2\cos y \sin y} = \frac{4x^3 \cos y}{x^4 \sin y + 2\cos y \sin y}$$

$$\boxed{\frac{dy}{dx} = \frac{4x^3 \cos y}{x^4 \sin y + 2\cos y \sin y}}$$

2. **(a)** For $y = \ln(\tan^{-1} x) + \sec^{-1}(e^x)$, find y'. Use my square bracket notation. You need not simplify your answer.

By the Chain Rule we have

$$\boxed{y' = \frac{1}{\tan^{-1}(x)}\left[\frac{1}{1+x^2}\right] + \frac{1}{|e^x|\sqrt{(e^x)^2-1}}\left[e^x\right],}$$

(b) Find the linearization of $f(x) = \tan^{-1}(x + e^{x^2})$ at $x = a = 0$.

The linearization of f at $x=a$ is $L(x) = f(a) + f'(a)(x-a)$.

We have $f'(x) = \dfrac{1}{1+(x+e^{x^2})^2}\left[1 + e^{x^2}[2x]\right] = \dfrac{1+2xe^{x^2}}{1+(x+e^{x^2})^2}$

so $f(a) = f(0) = \tan^{-1}((0)+e^{(0)^2}) = \tan^{-1}(1) = \pi/4$ and

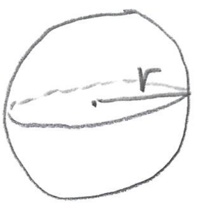
$f'(a) = f'(0) = \dfrac{1+2(0)e^{(0)^2}}{1+((0)+e^{(0)^2})^2} = \dfrac{1}{2}$.

Hence $L(x) = \dfrac{\pi}{4} + \dfrac{1}{2}(x-0) = \dfrac{1}{2}x + \dfrac{\pi}{4}$.

$$\boxed{L(x) = \frac{1}{2}x + \frac{\pi}{4}}$$

3, 4. A spherical balloon is inflated at a rate of 10 ft^3/min. How fast is the balloon's radius increasing when the radius is 9 ft? HINT: The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the surface area of a sphere is $S = 4\pi r^2$. Follow the 6 steps we used in Section 3.10.

① The picture is



where r is the radius of the balloon (as a function of time t).

② We know $\frac{dV}{dt} = 10$ ft^3/min where V is the volume of the balloon.

③ The question is: $\frac{dr}{dt} = ?$ when $r = 9$ ft and $\frac{dV}{dt} = 10$ ft^3/min.

④ The variables are related by the given formula $V = \frac{4}{3}\pi r^3$

⑤ Differentiating implicitly with respect to time t:

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] \text{ or } \frac{dV}{dt} = \frac{4}{3}\pi\left[3r^2\left[\frac{dr}{dt}\right]\right]$$

$$= 4\pi r^2 \frac{dr}{dt}$$

or $$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}.$$

⑥ Evaluate: WHEN $r = 9$ ft and $\frac{dV}{dt} = 10$ ft^3/min, we have

$$\frac{dr}{dt} = \frac{(10 \text{ ft}^3/\text{min})}{4\pi\,(9 \text{ ft})^2} = \frac{10}{4\pi(81)} \text{ ft/min}$$

$$= \frac{5}{162\pi} \text{ ft/min}.$$

$$\boxed{\frac{dr}{dt} = \frac{5}{162\pi} \text{ ft/min}}$$

5. Find the absolute maximum and minimum values of $f(x) = 2x^3 - 9x^2 - 1$ on the interval $[0, 4]$.

We follow the three steps of Section 4.1. First, $f'(x) = 6x^2 - 18x = 6x(x-3)$. Second, we find the critical points of f. Now $f'(x) = 6x(x-3) = 0$ implies that $x=0$ and $x=3$ are critical points. (Notice that there are no critical points where f' is undefined.) Third, we consider the values of f at the endpoints and critical points in $[0,4]$:

x	0	3	4
$f(x)$	$2(0)^3 - 9(0)^2 - 1$ $= -1$	$2(3)^3 - 9(3)^2 - 1$ $= 54 - 81 - 1 = -28$	$2(4)^3 - 9(4)^2 - 1$ $= 128 - 144 - 1 = -17$

So the maximum of f is -1 and occurs at $x = -1$, the minimum of f is -28 and occurs at $x = -28$.

$\boxed{\text{MAX is } -1 \quad \text{MIN is } -28}$

BONUS. (10 points) Use the linearization of $f(x) = \tan^{-1}(x + e^{x^2})$ at $x = a = 0$ from question #2(b) to *estimate* that value of f at $x = 0.1$.

We have from #2(b) that $L(x) = \frac{1}{2}x + \frac{\pi}{4}$, so since 0.1 is "close to" $a = 0$, then

$$f(0.1) \approx L(x) = \frac{1}{2}(0.1) + \frac{\pi}{4} = \frac{\pi}{4} + \frac{1}{20} = \frac{\pi}{4} + 0.05.$$

$$\boxed{\frac{\pi}{4} + \frac{1}{20}}$$