Honors Calculus 1 Test 4 — Fall 2011

NAME KEY E-NUMBER

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Indicate when you have used L'Hôpital's Rule and don't forget the "+C" when necessary, as we did in class. Your grade is based on the work you show and the arguments you make. If applicable, put your answer in the box provided. Each numbered problem is worth 12 points. **No calculators.**

1. Evaluate $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

$\boxed{\text{p 261 \# 16}}$

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} \overset{0/0}{=} \lim_{x\to 0}\left(\frac{\cos(x)-1}{3x^2}\right) \overset{0/0}{=} \lim_{x\to 0}\left(\frac{-\sin(x)}{6x}\right)$$

$$\overset{0/0}{=} \lim_{x\to 0}\left(\frac{-\cos(x)}{6}\right) = \frac{-\cos(0)}{6} = \frac{-1}{6}$$

$\boxed{-1/6}$

2. Evaluate $\lim_{x\to 0+} x^x$.

$\boxed{\text{p 261 \# 59}}$

Let $y = \lim_{x\to 0^+}(x^x)$. Then $\ln(y) = \ln\left(\lim_{x\to 0^+} x^x\right)$

$= \lim_{x\to 0^+}\left(\ln(x^x)\right)$ since natural log is continuous

$= \lim_{x\to 0^+}\left(x\ln(x)\right) = \lim_{x\to 0^+}\left(\frac{\ln(x)}{1/x}\right) \overset{0/0}{=} \lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)$

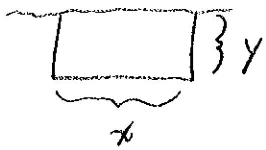
$= \lim_{x\to 0^+}\left(\frac{-x^2}{x}\right) = \lim_{x\to 0^+}(-x) = 0$,

$\boxed{1}$

SO, $e^{\ln(y)} = e^0$ and $y = 1$.

3,4. A rectangluar plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose? Use the 5 step method used in class.

p268 #7

① 

② The question is to maximize area $A = xy$

③ The amount of fencing is $x + 2y = 800m$, so $x = 800 - 2y$.

④ Area is $A = xy = (800-2y)\cdot y$
$= 800y - 2y^2 = A(y)$

⑤ Max $A(y) = 800y - 2y^2$ for $y \in [0, 400]$.

$A'(y) = 800 - 4y$, so set $A'(y) = 800 - 4y = 0 \Rightarrow y = 200$ is a critical point.

Consider

y	$A(y)$
0	0
200	160000 − 80000
400	0

So, MAX is 80,000 m^2 when $x = 400$ m and $y = 200$ m.

$\boxed{80{,}000 \ m^2}$

5. (a) Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2.

p277 #2

Well, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, where $f(x) = x^3 + 3x + 1$ and $f'(x) = 3x^2 + 3$.

So, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 0 - \dfrac{1}{3} = -\dfrac{1}{3}$,

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = \left(-\dfrac{1}{3}\right) - \dfrac{\left(-\frac{1}{3}\right)^3 + 3\left(-\frac{1}{3}\right) + 1}{3\left(-\frac{1}{3}\right)^2 + 3}$

$= -\dfrac{1}{3} - \dfrac{\left(-\frac{1}{27}\right)}{\left(\frac{10}{3}\right)} = -\dfrac{1}{3} + \dfrac{1}{90} = \dfrac{-29}{30}$

$\boxed{-29/30}$

5. **(b)** Evaluate $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$. Be sure to use the correct notation for your answer.

$\boxed{\begin{array}{l} p\,285 \\ \#31 \end{array}}$

$$\int \left(x^{-2} - x^2 - \frac{1}{3}\right) dx = -x^{-1} - \frac{1}{3}x^3 - \frac{1}{3}x + C$$

$$\boxed{-\frac{1}{x} - \frac{1}{3}x^3 - \frac{1}{3}x + C}$$

6. Give a complete definition of $\int_a^b f(x)\,dx$. Include definitions of *partition*, *norm* of the partition, and *Riemann sum*.

$\boxed{\begin{array}{l} p\,311, \\ 312, 314 \end{array}}$

A <u>partition</u> of $[a,b]$ is a set $P = \{x_0, x_1, \ldots, x_n\}$ where $a = x_0 < x_1 < \cdots < x_n = b$. Define $\Delta x_k = x_k - x_{k-1}$. The <u>norm</u> of a partition is $\|P\| = \max_{1 \le k \le n} \Delta x_k$. A <u>Riemann sum</u> is $S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k$ where $c_k \in [x_{k-1}, x_k]$.

The <u>definite integral</u> of f over $[a,b]$ is

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \left(\sum_{k=1}^{n} f(c_k)\,\Delta x_k \right).$$

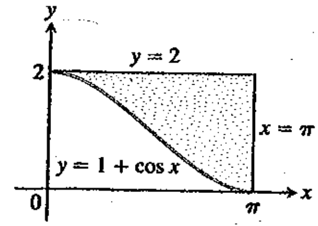
7. State the Fundamental Theorem of Calculus (both parts, with hypotheses).

$\boxed{\begin{array}{l} p\,327, \\ 328 \end{array}}$

<u>Part I</u> If f is continuous on $[a,b]$ then the function $F(x) = \int_a^x f(t)\,dt$ has a derivative at every point x in $[a,b]$ and $\frac{dF}{dx} = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$.

<u>Part II</u> If f is continuous at every point of $[a,b]$ and if F is any antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$



8. Find the area between the graphs of $y = 1 + \cos x$ and $y = 2$ for $x \in [0, \pi]$.

p 334 #61

Area is $A = (2)(\pi) - \int_0^\pi (1 + \cos(x))\,dx$

$= 2\pi - (x + \sin(x))\Big|_0^\pi = 2\pi - \{(\pi + \sin(\pi)) - (0 + \sin(0))\}$

$= 2\pi - \pi - 0 = \pi.$

$\boxed{\pi}$

Honors Bonus. Find dy/dx for $y = \int_{-1}^{x^{1/\pi}} \sin^{-1} t\, dt$. Explain your answer.

p 334 #56

$\frac{dy}{dx} = \frac{d}{dx}\left[\int_{-1}^{x^{1/\pi}} \sin^{-1}(t)\,dt\right] = \frac{d}{du}\left[\int_{-1}^{u} \sin^{-1}(t)\,dt\right] \cdot \left[\frac{du}{dx}\right]$ where $u = x^{1/\pi}$, by the Chain Rule

$= \sin^{-1}(u)\left[\frac{1}{\pi} x^{1/\pi - 1}\right]$ by F.T.C. II

$= \sin^{-1}\left(x^{1/\pi}\right) \frac{1}{\pi} x^{1/\pi - 1}$

$$\boxed{\frac{x^{1/\pi - 1} \sin^{-1}\left(x^{1/\pi}\right)}{\pi}}$$