

Calculus 1 Test 4 — Fall 2011

NAME KEY E-NUMBER _____

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*. Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Indicate when you have used L'Hôpital's Rule and don't forget the "+C" when necessary, as we did in class. Your grade is based on the work you show and the arguments you make. If applicable, put your answer in the box provided. Each numbered problem is worth 12 points. **No calculators.**

1. Evaluate $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$.

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#16

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{\sin(x) - x}{x^3} \right) &\stackrel{0/0}{=} \lim_{x \rightarrow 0} \left(\frac{\cos(x) - 1}{3x^2} \right) \stackrel{0/0}{=} \lim_{x \rightarrow 0} \left(\frac{-\sin(x)}{6x} \right) \\ &\stackrel{0/0}{=} \lim_{x \rightarrow 0} \left(\frac{-\cos(x)}{6} \right) = \frac{-\cos(0)}{6} = \frac{-1}{6} \end{aligned}$$

$-1/6$

2. Evaluate $\lim_{x \rightarrow 0^+} x^x$.

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#59

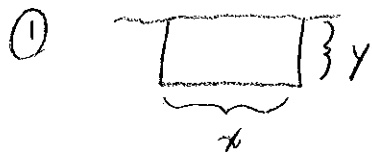
$$\begin{aligned} \text{Let } y &= \lim_{x \rightarrow 0^+} (x^x). \text{ Then } \ln(y) = \ln\left(\lim_{x \rightarrow 0^+} x^x\right) \\ &= \lim_{x \rightarrow 0^+} (\ln(x^x)) \text{ since natural log is continuous} \\ &= \lim_{x \rightarrow 0^+} (x \ln(x)) = \lim_{x \rightarrow 0^+} \left(\frac{\ln(x)}{1/x} \right) \stackrel{0/0}{=} \lim_{x \rightarrow 0^+} \left(\frac{1/x}{-1/x^2} \right) \\ &= \lim_{x \rightarrow 0^+} \left(\frac{-x^2}{x} \right) = \lim_{x \rightarrow 0^+} (-x) = 0. \end{aligned}$$

1

So, $e^{\ln(y)} = e^0$ and $y = 1$.

- 3,4. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose? Use the 5 step method used in class.

p268
#7



$$A'(y) = 800 - 4y, \text{ so set}$$

$$A'(y) = 800 - 4y = 0 \Rightarrow y = 200$$

is a critical point.

Consider

y	$A(y)$
0	0
200	160,000 - 80,000
400	0

so, MAX is 80,000 m²

when $x = 400$ m and $y = 200$ m.

80,000 m²

5. (a) Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2 .

Well, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, where $f(x) = x^3 + 3x + 1$ and $f'(x) = 3x^2 + 3$.

so, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{3} = -\frac{1}{3}$,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \left(-\frac{1}{3}\right) - \frac{\left(-\frac{1}{3}\right)^3 + 3\left(-\frac{1}{3}\right) + 1}{3\left(-\frac{1}{3}\right)^2 + 3}$$

-29/30

$$= -\frac{1}{3} - \frac{\left(-\frac{1}{27}\right)}{\left(\frac{10}{3}\right)} = -\frac{1}{3} + \frac{1}{90} = -\frac{29}{90}$$

5. (b) Evaluate $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3} \right) dx$. Be sure to use the correct notation for your answer.

$$\int \left(x^{-2} - x^2 - \frac{1}{3} \right) dx = -x^{-1} - \frac{1}{3} x^3 - \frac{1}{3} x + C$$

$$-\frac{1}{x} - \frac{1}{3} x^3 - \frac{1}{3} x + C$$

6. Give a complete definition of $\int_a^b f(x) dx$. Include definitions of *partition*, *norm* of the partition, and *Riemann sum*.

A partition of $[a, b]$ is a set $P = \{x_0, x_1, \dots, x_n\}$ where $a = x_0 < x_1 < \dots < x_n = b$. Define $\Delta x_k = x_k - x_{k-1}$. The norm of a partition is $\|P\| = \max_{1 \leq k \leq n} \Delta x_k$. A Riemann sum

is $S_n = \sum_{k=1}^n f(c_k) \Delta x_k$ where $c_k \in [x_{k-1}, x_k]$.

The definite integral of f over $[a, b]$ is

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \left(\sum_{k=1}^n f(c_k) \Delta x_k \right).$$

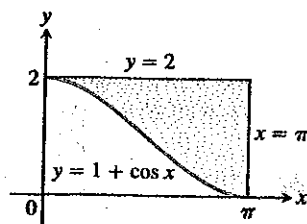
7. State the Fundamental Theorem of Calculus (both parts, with hypotheses).

Part I If f is continuous on $[a, b]$ then the function $F(x) = \int_a^x f(t) dt$ has a derivative at every point x in $[a, b]$ and $\frac{dF}{dx} = \frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x)$.

Part II If f is continuous at every point of $[a, b]$ and if F is any antiderivative of f on $[a, b]$ then

$$\int_a^b f(x) dx = F(b) - F(a).$$

8. Find the area between the graphs of $y = 1 + \cos x$ and $y = 2$ for $x \in [0, \pi]$.



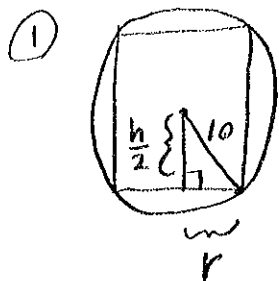
Area is $A = (2)(\pi) - \int_0^{\pi} (1 + \cos(x)) dx$

$$= 2\pi - \left(x + \sin(x) \right) \Big|_0^{\pi} = 2\pi - \{ (\pi + \sin(\pi)) - (0 + \sin(0)) \}$$

$$= 2\pi - \pi - 0 = \pi.$$

π

Bonus. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. Use the 5 step method used in class. HINT: The volume of a right circular cylinder is $V = \pi r^2 h$.



(2) The question is
MAX volume where
 $V = \pi r^2 h$

(3) We know by Pythagoras
 $\left(\frac{h}{2}\right)^2 + r^2 = (10)^2$
or $r^2 = 100 - \frac{h^2}{4}$

(4) Volume is
 $V = \pi r^2 h = \pi \left(100 - \frac{h^2}{4}\right) h = V(h).$

(5) MAX $V(h)$ for $h \in [0, 20]$.

Well, $V'(h) = 100\pi - \frac{3\pi h^2}{4}$ and we

set $V'(h) = 100\pi - \frac{3\pi h^2}{4} \equiv 0 \Rightarrow$

$h^2 = \frac{400}{3} \Rightarrow h = \pm \frac{20}{\sqrt{3}}$ is a

critical point. Consider:

h	$V(h)$
0	0
$20/\sqrt{3}$	$\frac{4000}{3\sqrt{3}}\pi$
20	0

$$\frac{4000\pi}{3\sqrt{3}} \text{ cm}^3$$

So, V is a MAX of
 $\frac{4000\pi}{3\sqrt{3}} \text{ cm}^3$ when

$h = \frac{20}{\sqrt{3}} \text{ cm}$ and

$r = 10\sqrt{2}/\sqrt{3} \text{ cm}.$