Calculus 1 Test 4 — Fall 2012

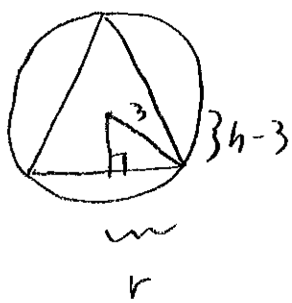
NAME KEY E-NUMBER ____________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Don't forget the "$+C$" when necessary, as we did in class. Your grade is based on the work you show and the arguments you make. If applicable, put your answer in the box provided. Each numbered problem is worth 12 points. **No calculators.**

p 269 #12

~~1.~~ **2.** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3. HINT: The volume of a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$. Use the Pythagorean Theorem to find a relationship between the radius of the circle, the radius of the cone, and part of the height of the cone.

(1)



(2) The question is "MAX $V = \frac{1}{3}\pi r^2 h$."

(3) We know $r^2 + (h-3)^2 = (3)^2$ or $r^2 = 9-(h-3)^2$.

(4) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(9-(h-3)^2\right)h = \frac{1}{3}\pi(6h-h^2)h$

$= \frac{1}{3}\pi(6h^2-h^3) = V(h)$

(5) MAX $V(h) = \frac{1}{3}\pi(6h^2-h^3)$ on $h \in [3,6]$:

$V'(h) = \frac{\pi}{3}(12h-3h^2) = \frac{\pi}{3}h(12-3h)$. So $h=0$ and $h=4$ are critical points. Consider

h	$V(h)$
3	$\frac{\pi}{3}\left(6(3)^2-(3)^3\right) = 9\pi$
4	$\frac{\pi}{3}\left(6(4)^2-(4)^3\right) = \frac{32\pi}{3}$
6	0

$\boxed{\frac{32\pi}{3}}$

So MAX volume is $\frac{32\pi}{3}$

3. Use Newton's method to estimate the one real solution of $x^3+3x+1=0$. Start with $x_0=0$ and then find x_2.

p277 #2

Well, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ where $f(x) = x^3+3x+1$ and $f'(x) = 3x^2+3$.

So $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 0 - \dfrac{1}{3} = \dfrac{-1}{3}$.

Next, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = \left(-\dfrac{1}{3}\right) - \dfrac{\left(-\frac{1}{3}\right)^3 + 3\left(\frac{-1}{3}\right)+1}{3\left(-\frac{1}{3}\right)^2+3}$

$= \left(-\dfrac{1}{3}\right) - \dfrac{\left(-\frac{1}{27}\right)}{\left(\frac{10}{3}\right)} = -\dfrac{1}{3} + \dfrac{1}{90} = \dfrac{-29}{90}$ $\boxed{\dfrac{-29}{90}}$

p286 #57

4. Evaluate $\int(\sin 2x - \csc^2 x)\,dx$. Be sure to use the correct notation for your answer.

$\int(\sin(2x) - \csc^2(x))\,dx = \int \sin(2x)\,dx - \int \csc^2(x)\,dx$

$= -\frac{1}{2}\cos(2x) - (-\cot(x)) + C$

$= -\frac{1}{2}\cos(2x) + \cot(x) + C$

$\boxed{-\frac{1}{2}\cos(2x) + \cot(x) + C}$

5. Give a complete definition of $\int_a^b f(x)\,dx$. Include definitions of *partition*, *norm* of the partition, and *Riemann sum*.

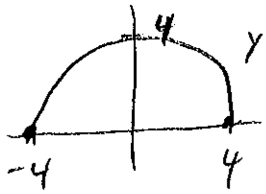
p311, 312, 314

A <u>partition</u> of $[a,b]$ is a set $P = \{x_0, x_1, \dots, x_n\}$ where $a = x_0 < x_1 < \dots < x_n = b$. Define $\Delta x_k = x_k - x_{k-1}$. The <u>norm</u> of partition P is $\|P\| = \max\limits_{1\le k\le n} \Delta x_k$. A <u>Riemann sum</u> is $S_n = \sum\limits_{k=1}^{n} f(c_k)\Delta x_k$ where $c_k \in [x_{k-1}, x_k]$. The <u>definite integral</u> of f over $[a,b]$ is $\int_a^b f(x)\,dx = \lim\limits_{\|P\|\to 0}\left(\sum\limits_{k=1}^{n} f(c_k)\Delta x_k\right)$.

p322 #18

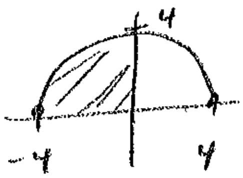
6. Recall that definite integrals of nonnegative functions represent areas bounded below curves and above the x-axis over the given interval of integration. Use areas to evaluate $\int_{-4}^{0} \sqrt{16-x^2}\,dx$. HINT: The graph of $y = \sqrt{16-x^2}$ is a semicircle.

Well, the graph is



So $\int_{-4}^{0} \sqrt{16-x^2}\,dx$

is:



and $\int_{-4}^{0} \sqrt{16-x^2}\,dx = \frac{1}{4}\pi(4)^2 = 4\pi$

$\boxed{4\pi}$

7. State the Fundamental Theorem of Calculus (both parts, with hypotheses).

p327, 328

Part I If f is continuous on $[a,b]$ then the function $F(x) = \int_a^x f(t)\,dt$ has a derivative at every point x in $[a,b]$ and $\frac{dF}{dx} = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$.

Part II If f is continuous at every point of $[a,b]$ and if F is any antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

p333 #30

8. Evaluate $\int_1^2 \left(\frac{1}{x} - e^{-x}\right) dx.$ $= \int_1^2 (x^{-1} - e^{-x})\,dx = \left(\ln|x| - (-e^{-x})\right)\Big|_1^2$

$= (\ln(2) + e^{-2}) - (\ln(1) + e^{-1}) = \ln(2) + e^{-2} - e^{-1}$.

$\boxed{\ln(2) + e^{-2} + e^{-1}}$

$\boxed{p\,323\ \#65 \text{ with } a=0, b=1}$

Bonus. Use a regular partition (in which each subinterval is of the same length) and $c_k = x_k$ to set up a Riemann sum for the integral $\int_0^1 x^2\,dx$. Evaluate the integral by letting $n \to \infty$ (in which case $\|P\| \to 0$). HINT: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

With a regular partition, $\Delta x_k = \Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$ and so $\|P\| = \frac{1}{n}$. Also, $x_k = a + k\Delta x = 0 + k\left(\frac{1}{n}\right) = \frac{k}{n}$.

Choose $c_k \in [x_{k-1}, x_k]$ as $c_k = x_k = \frac{k}{n}$. Then a Riemann sum is

$$\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right)$$

$$= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \left(\frac{1}{n}\right) = \frac{1}{n^3}\sum_{k=1}^{n} (k^2) = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.$$

So $\lim_{\|P\|\to 0}\left(\sum_{k=1}^{n} f(c_k)\,\Delta x_k\right) = \lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)}{6n^3}\right) = \lim_{n\to\infty}\left(\frac{2n^3+3n^2+n}{6n^3}\right)$

$\overset{\infty/\infty}{=} \lim_{n\to\infty}\left(\frac{6n^2+6n}{18n^2}\right) \overset{\infty/\infty}{=} \lim_{n\to\infty}\left(\frac{12n+6}{36n}\right)$

$\overset{\infty/\infty}{=} \lim_{n\to\infty}\left(\frac{12}{36}\right) = \frac{12}{36} = \frac{1}{3}$

$$\boxed{\frac{1}{3}}$$