Calculus 1 Test 4 (Online) — Fall 2020

NAME KEY STUDENT NUMBER ________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols. **Write in complete sentences!** Your grade is based on the work you show and the arguments you make. If applicable, put your final answer in the box provided. Each numbered problem is worth 20 points.

You are not to collaborate with anyone during the test! When you are done, scan your solutions and put them in the Dropbox in D2L. If you have trouble with D2L (which I wouldn't expect), then e-mail me your solutions at `gardnerr@etsu.edu` (but please try to get your solutions submitted in D2L). I prefer PDFs, but if things aren't working then go ahead and just submit photographs of your solutions. The test is due by 11:15 but you can still submit to DropBox until 11:30.

1. (a) Find the value c that satisfies $\dfrac{f(b)-f(a)}{b-a} = f'(c)$ in the conclusion of the Mean Value Theorem for $f(x) = e^x$ on the interval $[0,1]$.

Since $f'(x) = e^x$, we need $e^c = \dfrac{e^1 - e^0}{1-0} = e-1$, or $\ln(e^c) = \ln(e-1)$ or $c = \ln(e-1)$.

$$\boxed{\ln(e-1)}$$

(b) Use L'Hopital's Rule to evaluate $\lim\limits_{x\to 0} \dfrac{\sin(5x)-5x}{x^3}$. Whenever you use L'Hopital's Rule, put the indeterminate form of the limit over the equal sign.

$$\lim_{x\to 0} \frac{\sin(5x)-5x}{x^3} \overset{0/0}{=} \lim_{x\to 0} \frac{5\cos(5x)-5}{3x^2} \overset{0/0}{=} \lim_{x\to 0} \frac{-25\sin(5x)}{6x}$$

$$\overset{0/0}{=} \lim_{x\to 0} \frac{-125\cos(5x)}{6} = \frac{-125\cos(5(0))}{6}$$

$$= \frac{-125(1)}{6} = \frac{-125}{6}.$$

$$\boxed{\frac{-125}{6}}$$

2. Consider $f(x) = x^4 - 8x^3 + 18x^2$. Find the critical points, the intervals on which f is increasing or decreasing, and the local extrema.

Since $f'(x) = 4x^3 - 24x^2 + 36x = 4x(x^2 - 6x + 9) = 4x(x-3)^2$
then $x=0$ and $x=3$ are critical points since $f'=0$ there
so we consider

	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
TEST VALUE k	-1	1	4
$f'(k)$	$4(-1)((-1)-3)^2 = -64$	$4(1)((1)-3)^2$ 16	$4(4)((4)-3)^2 = 16$
$f'(x)$	$-$	$+$	$+$
$f(x)$	DEC	INC	INC

So f is DEC on $(-\infty, 0]$ and f is IN on $[0, \infty)$.
By the First Derivative Test for Local Extrema (Theorem 4.3.A), f has a local MIN at $x=0$ of $f(0)=0$ and f has no local MAX.

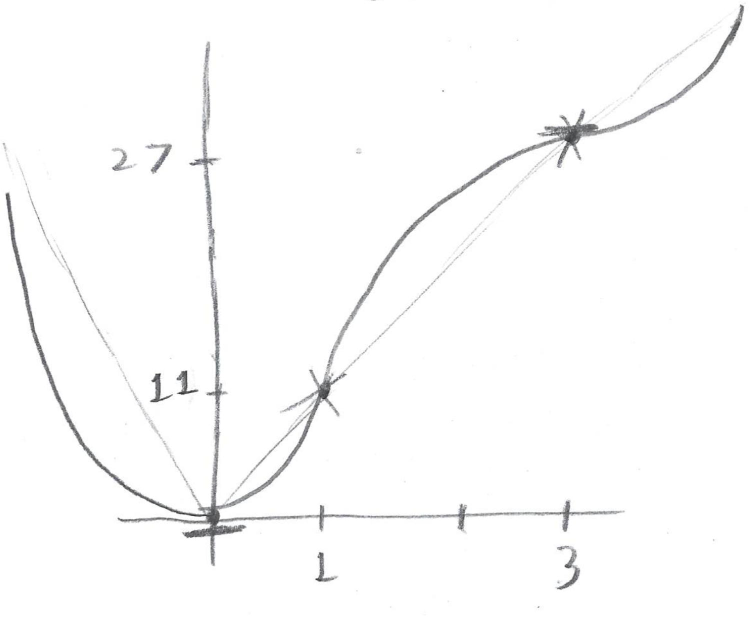
Critical points: $0, 3$
INC on $(-\infty, 0]$
DEC on $[0, \infty)$
local MIN at $x=0$ of 0,
no local MAX.

3. Consider $f(x) = x^4 - 8x^3 + 18x^2$ again. Find the points of inflection and the intervals on which f is concave up or concave down. Use this information and the information from problem #3 to graph $y = f(x)$.

Since $f'(x) = 12x^2 - 48x + 36 = 12(x^2 - 4x + 3) = 12(x-1)(x-3)$ then the potential points of inflection are $x = 1$ and $x = 3$. So we consider

	$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$
TEST VALUE k	0	2	4
$f''(k)$	$12((0)-1)((0)-3) = 36$	$12((2)-1)((2)-3) = -12$	$12((4)-1)((4)-3) = 36$
$f''(x)$	$+$	$-$	$+$
$f(x)$	CU	CD	CU

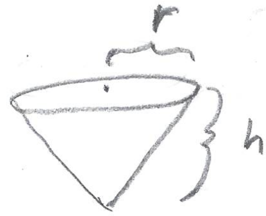
So $x = 1$ and $x = 3$ are points of inflection. f is CU on $(-\infty, 1) \cup (3, \infty)$ and f is CD on $(1, 3)$.
Notice that $f(1) = (1)^4 - 8(1)^3 + 18(1)^2 = 11$ and
$f(3) = (3)^4 - 8(3)^3 + 18(3)^2 = 81 - 8(27) + 18(9)$
$= 81 - 216 + 162 = 27$. The graph is



pts of inflection 1, 3
CU on $(-\infty, 1) \cup (3, \infty)$
CD on $(1, 3)$

4, 5. A paper cup having the shape of a right circular cone is to be made from a flat sheet of paper. If the volume of the cup is to be 36π in.^{3}, what are the dimensions that require the least amount of paper. HINT: The volume of a cone of base radius r and height h is $V = \frac{1}{3}\pi r^2 h$ and the area of the slanted surface is $S = \pi r\sqrt{r^2+h^2}$. Follow the steps of Section 4.6. HINT: Write A as a function of h.

(2) and (3) The cone is:



We let r be the base radius and h be the height. The area of the slanted surface is $S = \pi r\sqrt{r^2+h^2}$.

The volume $V = \frac{1}{3}\pi r^2 h = 36\pi$ in.^{3}, so $r = \sqrt{\frac{108}{h}}$.

(4) Surface area is $S = \pi r\sqrt{r^2+h^2} = \pi\sqrt{\frac{108}{h}}\sqrt{\frac{108}{h}+h^2}$

$= \pi\left(\frac{(108)^2}{h^2} + 108h\right)^{1/2} = S(h)$. The question is to find the dimensions that minimize $S(h)$ (and hence minimize the amount of paper) for $h \in (0,\infty)$.

(5) We have $S'(h) = \frac{\pi}{2}\left(\frac{(108)^2}{h^2} + 108h\right)^{-1/2}\left[\frac{-2(108)^2}{h^3} + 108\right]$

$= \dfrac{\pi\left(108 - 2(108)^2/h^3\right)}{2\sqrt{(108)^2/h^2 + 108h}}$. So we have a critical point

if $108 - 2(108)^2/h^3 = 0$ or $\frac{1}{h^3} = \frac{108}{2(108)^2} = \frac{1}{216}$ or

$h^3 = 216$ or $h = 6$ in. Notice that when $h = 6$ in. then

$r = \sqrt{\frac{108}{6}} = \sqrt{18} = 3\sqrt{2}$ in.

We consider where S is INC/DES as follows...

$$\boxed{\begin{array}{l} h = 6 \text{ in} \\ r = 3\sqrt{2} \text{ in} \end{array}}$$

Consider

	$(0, 6)$	$(6, \infty)$
TEST VALUE h	1	10
$S'(h)$	$\dfrac{\pi\left(108 - 2(108)^2/(1)^3\right)}{2\sqrt{(108)^2/(1)^2 + 108(1)}}$	$\dfrac{\pi\left(108 - 2(108)/(10)^3\right)}{2\sqrt{(108)^2/(10)^2 + 108(10)}}$
$S'(h)$	$-$	$+$
$S(h)$	DEC	INC

So by the First Derivative Test for Local Extrema (Theorem 4.3.A), S has a local MIN at $h = 6$ in. Since $h = 6$ is the only critical point in $(0, \infty)$ then the absolute MIN of S also occurs at $h = 6$ in. So the surface area (and hence the amount of paper used) is minimized when $h = 6$ in and $r = 3\sqrt{2}$ in.

Notice that this MIN is

$$S(6) = \pi\sqrt{\frac{(108)^2}{(6)^2} + 108(6)} = \pi\sqrt{(18)^2 + 648}$$

$$= \pi\sqrt{972} = \pi\sqrt{3(4)(81)} = 18\pi\sqrt{3} \text{ in.}^2$$

BONUS. (10 points) In Exercise 2.5.57, the Mean Value Theorem is used to show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$. Use Newton's Method to estimate one of these solutions by starting with $x_0 = 0$ and then finding x_2.

With $f(x) = x^3 - 15x + 1$, we have $f'(x) = 3x^2 - 15$.

Newton's Method gives $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, so with

$x_0 = 0$ we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = (0) - \frac{(0)^3 - 15(0) + 1}{3(0)^2 - 15} = \frac{1}{15}.$$

Then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\boxed{\frac{3373}{50{,}580} \approx 0.067}$$

$$= \left(\frac{1}{15}\right) - \frac{\left(\frac{1}{15}\right)^3 - 15\left(\frac{1}{15}\right) + 1}{3\left(\frac{1}{15}\right)^2 - 15}$$

$$= \frac{1}{15} - \frac{\frac{1}{(15)^3} - 1 + 1}{\frac{3}{(15)^2} - 15} = \frac{1}{15} - \left(\frac{\frac{1}{(15)^3}}{\frac{3}{(15)^2} - 15}\right)\left(\frac{(15)^3}{(15)^3}\right)$$

$$= \frac{1}{15} - \frac{1}{3(15) - (15)^4} = \frac{1}{15} - \frac{1}{45 - 50{,}625}$$

$$= \frac{1}{15} + \frac{1}{50{,}580} = \frac{3372 + 1}{50{,}580}$$

$$= \frac{3373}{50{,}580} \approx 0.067$$