Calculus 1 Test 5 (Online) — Fall 2020

NAME KEY STUDENT NUMBER

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols and respect the notation for indefinite integrals (they are *sets*!). **Write in complete sentences!** Your grade is based on the work you show and the arguments you make. If applicable, put your final answer in the box provided. Each numbered problem is worth 20 points.

You are not to collaborate with anyone during the test! When you are done, scan your solutions and put them in the Dropbox in D2L. If you have trouble with D2L (which I wouldn't expect), then e-mail me your solutions at `gardnerr@etsu.edu` (but please try to get your solutions submitted in D2L). I prefer PDFs, but if things aren't working then go ahead and just submit photographs of your solutions. The test is due by 11:15 but you can still submit to DropBox until 11:30.

1. Solve the initial value problem: $\dfrac{dy}{dx} = 6x^2 + 2x - 1$ and $y(1) = 0$.

We have $y \in \int \frac{dy}{dx}\,dx = \int (6x^2+2x-1)\,dx = 6\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) - x + C$

$= 2x^3 + x^2 - x + C.$

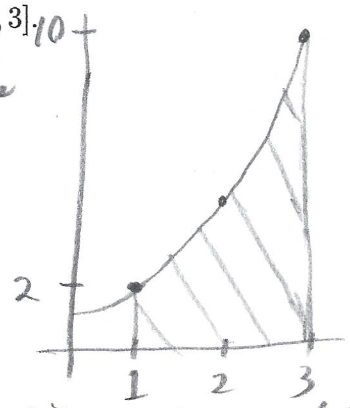
So $y = 2x^3 + x^2 - x + k$ for some constant k.

Since $y(1) = 0$ we need $y(1) = 2(1)^3 + (1)^2 - (1) + k = 0$ or $2 + k = 0$ or $k = -2$. Hence $y = 2x^3 + x^2 - x - 2$.

$$\boxed{y = 2x^3 + x^2 - x - 2}$$

2. Consider $f(x) = x^2 + 1$ on the interval $[a, b] = [1, 3]$. Partition the interval $[1, 3]$ into $n = 4$ subintervals and set up a lower Riemann sum to approximate the area under the graph of $y = f(x)$ and above the x-axis over the interval $[1, 3]$.

The area we are interested in is



with $n=4$ subintervals of $[1,3]$, the subintervals are $[1, 3/2]$, $[3/2, 2]$, $[2, 5/2]$, $[5/2, 3]$. Since f is an increasing function, then a lower Riemann sum is calculated using left-hand endpoints (i.e., $c_k = x_{k-1}$).

So the Riemann sum is

$$\sum_{k=1}^{4} f(c_k)\,\Delta x_k = f(1)\Delta x_1 + f(3/2)\Delta x_2 + f(2)\Delta x_3 + f(5/2)\Delta x_4$$

$$= \left(f(1) + f(3/2) + f(2) + f(5/2)\right)\left(\tfrac{1}{2}\right) \text{ since each } \Delta x_k = \tfrac{1}{2}$$

$$= \left\{\left((1)^2+1\right) + \left(\left(\tfrac{3}{2}\right)^2+1\right) + \left((2)^2+1\right) + \left(\left(\tfrac{5}{2}\right)^2+1\right)\right\}\left(\tfrac{1}{2}\right)$$

$$= \left\{(2) + \left(\tfrac{9}{4} + \tfrac{4}{4}\right) + (5) + \left(\tfrac{25}{4} + \tfrac{4}{4}\right)\right\}\left(\tfrac{1}{2}\right)$$

$$= \left(2 + \tfrac{13}{4} + 5 + \tfrac{29}{4}\right)\left(\tfrac{1}{2}\right) = \left(7 + \tfrac{21}{2}\right)\left(\tfrac{1}{2}\right) = \left(\tfrac{35}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{35}{4}$$

$$\boxed{\sum_{k=1}^{4} f(c_k)\,\Delta x_k = \frac{35}{4} \quad \text{where } c_k = x_{k-1} \text{ and } \Delta x = \Delta x_k = \frac{1}{2}}$$

3. For the function $f(x) = x^2 + 1$, find a formula for the Riemann sum obtained by dividing the interval $[a, b] = [0, 1]$ into n equal subintervals and using the right-hand endpoint for each c_k. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[0, 1]$.

With $f(x) = x^2+1$, f is continuous on $[0,1]$ and so is integrable by "Integrability of Continuous Functions" (Theorem 5.1). Therefore we can use an equal-width partition P to determine a Riemann sum, where $\|P\| = 1/n$. Then as $n \to \infty$, $\|P\| \to \infty$. We have $\Delta x_k = \Delta x = (b-a)/n = 1/n$, $x_k = a + k(b-a)/n = 0 + k(1/n) = k/n$, and $c_k = x_k = k/n$. The Riemann sum is then

$$\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n}\left(\left(\frac{k}{n}\right)^2 + 1\right)\left(\frac{1}{n}\right) = \sum_{k=1}^{n}\left(\frac{k^2}{n^3} + \frac{1}{n}\right)$$

$$= \frac{1}{n^3}\left(\sum_{k=1}^{n} k^2\right) + \frac{1}{n}\left(\sum_{k=1}^{n} 1\right) = \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{n}(n)$$

$$= \frac{(n+1)(2n+1)}{6n^2} + 1 = \frac{2n^2+3n+1}{6n^2} + 1.$$

Then $$\lim_{\|P\|\to 0}\left(\sum_{k=1}^{n} f(c_k)\,\Delta x_k\right) = \lim_{n\to\infty}\left(\frac{2n^2+3n+1}{6n^2} + 1\right)$$

$$= \lim_{n\to\infty}\left(\frac{2n^2+3n+1}{6n^2}\right)\left(\frac{1/n^2}{1/n^2}\right) + 1 = \lim_{n\to\infty}\left(\frac{2n^2/n^2 + 3n/n^2 + 1/n^2}{6n^2/n^2}\right) + 1$$

$$= \lim_{n\to\infty}\left(\frac{2 + 3(1/n) + (1/n)^2}{6}\right) + 1$$

$$= \frac{2 + 3\lim_{n\to\infty}(1/n) + \left(\lim_{n\to\infty} 1/n\right)^2}{6} + 1$$

$$= \frac{2 + 3(0) + (0)^2}{6} + 1 = 1 + \frac{2}{6} = 1 + \frac{1}{3} = \frac{4}{3}$$

$$\boxed{\begin{array}{l}\text{Riemann Sum: } \dfrac{2n^2+3n+1}{6n^2} + 1 \\ \text{The limit of the Riemann Sum is } \dfrac{4}{3}.\end{array}}$$

4. (a) For $y = \int_1^{3x^2} e^{t^2}\, dt$, find dy/dx using the Fundamental Theorem of Calculus, Part 1.

Let $u = 3x^2$. Then by the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}\left[\int_1^u e^{t^2}\,dt\right] \frac{d}{dx}[3x^2]$$

$$= e^{u^2}[6x] = e^{(3x^2)^2}[6x]$$ by the Fundamental Theorem of Calculus, Part 1

$$= 6xe^{9x^4}.$$

$$\boxed{6xe^{9x^4}}$$

(b) Evaluate $\int_0^{\pi/2} \sin 3x\, dx$ using the Fundamental Theorem of Calculus, Part 2.

An antiderivative of $f(x) = \sin(3x)$ is $F(x) = \frac{-1}{3}\cos(3x)$, so by the Fundamental Theorem of Calculus, Part 2:

$$\int_0^{\pi/2} \sin(3x)\,dx = \frac{-1}{3}\cos(3x)\Big|_0^{\pi/2} = \frac{-1}{3}\cos\left(3\left(\frac{\pi}{2}\right)\right) - \frac{-1}{3}\cos(3(0$$

$$= \frac{-1}{3}\cos\left(\frac{3\pi}{2}\right) + \frac{1}{3}\cos(0) = \frac{-1}{3}(0) + \frac{1}{3}(1) = \frac{1}{3}$$

$$\boxed{\int_0^{\pi/2} \sin(3x)\,dx = \frac{1}{3}}$$

5. Find the area under the curve $y = f(x) = \sec x \tan x + 2$ for $x \in [-\pi/4, \pi/4]$. Notice that this function is nonnegative over the given interval.

Since the function is given to be nonnegative, then the area under $y = f(x)$ is given by the definite integral

$$A = \int_a^b f(x)\,dx = \int_{-\pi/4}^{\pi/4} (\sec x \tan x + 2)\,dx$$

$$= \left(\sec(x) + 2x\right)\Big|_{-\pi/4}^{\pi/4}$$

$$= \left(\sec\left(\tfrac{\pi}{4}\right) + 2\left(\tfrac{\pi}{4}\right)\right) - \left(\sec\left(\tfrac{-\pi}{4}\right) + 2\left(\tfrac{-\pi}{4}\right)\right)$$

$$= \left(\left(\sqrt{2}\right) + \tfrac{\pi}{2}\right) - \left(\left(\sqrt{2}\right) - \tfrac{\pi}{2}\right) = \pi.$$

$$\boxed{\pi}$$

BONUS. (10 points) Explain why $\int \ln x \, dx = x \ln x - x + C$.

Since the indefinite integral $\int \ln x \, dx$ is the <u>set</u> of all antiderivatives of $\ln x$, let $y \in \int \ln x \, dx$. Then we have $y = x \ln x - x + k$ for some constant k. We just need to confirm that $y' = \frac{dy}{dx} = \ln x$.

We have

$$y' = \frac{dy}{dx} = \frac{d}{dx}[x \ln x - x + k]$$

$$= [1](\ln x) + (x)\left[\frac{1}{x}\right] - 1 + 0$$

$$= \ln x + 1 - 1 = \ln x.$$

Since $\ln x$ is the integrand in $\int \ln x \, dx$, then we have that $\int \ln x \, dx = x \ln x - x + C$ ...

or, if you like, the "most general antiderivative" of $\ln x$ is $x \ln x - x + C$. $\square$