

Honors Calculus 1 Test 1 — Fall 2011

NAME KEY STUDENT NUMBER _____

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*. Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols (such as limits). Your grade is based on the work you show and the arguments you make. If applicable, put your answer in the box provided. Each numbered problem is worth 12 points. No calculators.

1. (a) Find the slope of the parabola $y = x^2$ at the point $P = (2, 4)$. (b) Write an equation for the tangent to the parabola at this point. **Hint:** Choose a second point $Q = (2 + h, (2 + h)^2)$ on the curve (where $h \neq 0$) and compute the slope of the secant line PQ . Guess what happens to the slope of the secant line when h is close to 0.

① The slope of the secant line through PQ is $\frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{(2+h) - 4} = \frac{4 + 4h + h^2 - 4}{2 + h - 2}$
 $= \frac{4h + h^2}{h} = \frac{h(4+h)}{h} = 4 + h$ since $h \neq 0$. When h is close to 0, the

slope is close to 4. GUESS: $m = 4$.

② The tangent line is $y - y_1 = m(x - x_1)$
 $y - 4 = 4(x - 2)$

$$y = 4x - 4$$

① $m = 4$

② $y = 4x - 4$

2. Evaluate $\lim_{y \rightarrow -2} \frac{y+2}{y^2+5y+6}$, if it exists.

$y \rightarrow -2$

$\lim_{y \rightarrow -2} \left(\frac{y+2}{y^2+5y+6} \right) \stackrel{F}{=} \lim_{y \rightarrow -2} \frac{y+2}{(y+2)(y+3)} \stackrel{c}{=} \lim_{y \rightarrow -2} \left(\frac{1}{y+3} \right) \stackrel{s}{=} \frac{1}{(-2)+3} = \boxed{1}$

1

p. 77

3. State the precise (or formal) definition of $\lim_{x \rightarrow x_0} f(x) = L$ (the definition with the ϵ and δ in it).

Let $f(x)$ be defined on an open interval about x_0 except possibly at x_0 itself. We say that $f(x)$ approaches limit L as x approaches x_0 and write $\lim_{x \rightarrow x_0} f(x) = L$ if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x , $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$.

p 91
#3

4. Consider $f(x) = \begin{cases} 3 - x & \text{if } x \in (-\infty, 2) \\ \frac{x}{2} + 1 & \text{if } x \in (2, \infty) \end{cases}$. Evaluate $\lim_{x \rightarrow 2} f(x)$, if it exists. Give a clear explanation of your answer!

Well, $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (3 - x) = 3 - (2) = 1$ and

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \left(\frac{x}{2} + 1 \right) = \frac{(2)}{2} + 1 = 2.$$

So, $\lim_{x \rightarrow 2} f(x)$ does not exist since $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$.

Does not exist.

5. Evaluate $\lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cos x}$, if it exists.

p 92
#29

$$\lim_{x \rightarrow 0} \frac{x + x \cos(x)}{\sin(x) \cos(x)} = \lim_{x \rightarrow 0} \frac{x(1 + \cos(x))}{\sin(x) \cos(x)} = \lim_{x \rightarrow 0} \frac{x}{\sin(x)} \lim_{x \rightarrow 0} \left(\frac{1 + \cos(x)}{\cos(x)} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{\frac{\sin(x)}{x}} \right) \lim_{x \rightarrow 0} \left(\frac{1 + \cos(x)}{\cos(x)} \right) = \frac{1}{\lim_{x \rightarrow 0} \left(\frac{\sin(x)}{x} \right)} \lim_{x \rightarrow 0} \left(\frac{1 + \cos(x)}{\cos(x)} \right)$$

$$= \frac{1}{(1)} \left(\frac{1 + \cos(0)}{\cos(0)} \right) = \frac{1 + (1)}{(1)} = 2$$

↑
since $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$

2

6. Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, find the extension and prove that the extension is continuous at $x = 2$.

Well, $\lim_{x \rightarrow 2} \left(\frac{x^2 + x - 6}{x^2 - 4} \right) \stackrel{f}{=} \lim_{x \rightarrow 2} \frac{(x-2)(x+3)}{(x-2)(x+2)} \stackrel{c}{=} \lim_{x \rightarrow 2} \left(\frac{x+3}{x+2} \right) \stackrel{5}{=} \frac{(2)+3}{(2)+2} = \frac{5}{4}$

and so $f(x)$ has a removable discontinuity at $x = 2$.

Define $g(x) = \frac{x+3}{x+2}$. Then $g(x) = f(x)$ for $x \neq 2$ and

$$\lim_{x \rightarrow 2} g(x) = \lim_{x \rightarrow 2} \left(\frac{x+3}{x+2} \right) = \frac{(2)+3}{(2)+2} = \frac{5}{4} = g(2), \text{ so } g \text{ is}$$

continuous at $x = 2$ and hence g is the continuous extension of f .

$$g(x) = \frac{x+3}{x+2}$$

7. Evaluate $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$.

$$\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}} = \lim_{x \rightarrow -\infty} \left(\frac{4 - 3x^3}{\sqrt{x^6 + 9}} \right) \left(\frac{1/x^3}{1/x^3} \right) = \lim_{x \rightarrow -\infty} \frac{\left(\frac{4}{x^3} - \frac{3x^3}{x^3} \right)}{\left(\frac{\sqrt{x^6 + 9}}{x^3} \right)}$$

$$= \lim_{x \rightarrow -\infty} \frac{\left(\frac{4}{x^3} - \frac{3x^3}{x^3} \right)}{\frac{\sqrt{x^6 + 9}}{x^3}} = \lim_{x \rightarrow -\infty} \frac{\left(\frac{4}{x^3} - 3 \right)}{\frac{\sqrt{1 + \frac{9}{x^6}}}{1}} = \frac{4 \left(\lim_{x \rightarrow -\infty} \frac{1}{x} \right)^3 - 3}{\sqrt{1 + 9 \left(\lim_{x \rightarrow -\infty} \frac{1}{x} \right)^6}}$$

$$= \frac{4(0)^3 - 3}{\sqrt{1 + 9(0)^6}} = \frac{-3}{1} = -3$$

$$-3$$

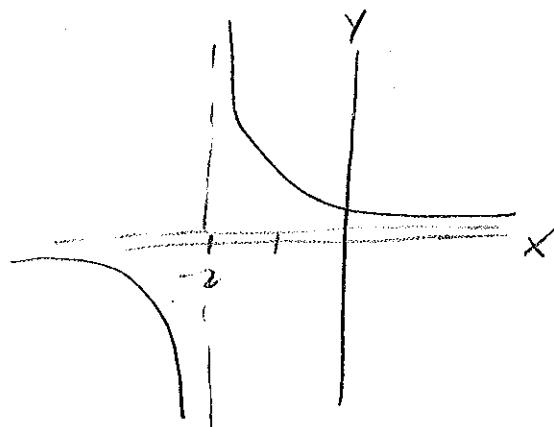
p115
#65

8. Consider $f(x) = \frac{1}{2x+4}$. Then $\lim_{x \rightarrow \pm\infty} f(x) = 0$. Use limits to find the vertical asymptotes of f and graph $y = f(x)$ in such a way as to reflect the asymptotes.

We check $f(x)$ for V.A. at $x = -2$ (where the denominator is 0):

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} \left(\frac{1}{2x+4} \right) \frac{(-)}{(-)} = -\infty$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} \left(\frac{1}{2x+4} \right) \frac{(+)}{(+)} = +\infty.$$



So $x = -2$ is a V.A. of f . Also,

since $\lim_{x \rightarrow \pm\infty} f(x) = 0$, then $y = 0$

is a H.A. of f .

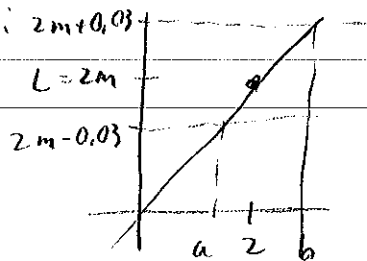
V.A. at $x = -2$

H.A. at $y = 0$

p83
#27

Honors Bonus. Consider $f(x) = mx$ where $m > 0$, $L = 2m$, $x_0 = 2$, and $\epsilon = 0.03$. Find $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$, the inequality $|f(x) - L| < \epsilon$ holds.

Well, the graph is:



$$\text{Now } f(a) = ma = 2m - 0.03 \Rightarrow a = 2 - \frac{0.03}{m}$$

$$f(b) = mb = 2m + 0.03 \Rightarrow b = 2 + \frac{0.03}{m}$$

$$\text{So } \delta = \min \{ 2 - a, b - 2 \} = \min \left\{ \frac{0.03}{m}, \frac{0.03}{m} \right\}$$

$$= \frac{0.03}{m}$$

$$\delta = \frac{0.03}{m}$$