Honors Calculus 1 Test 2 — Fall 2010

NAME KEY STUDENT NUMBER ________________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols. If applicable, put your answer in the box provided. Each numbered problem is worth 12 points and each bonus problem is worth 10 points. **Put your calculators away!!! This is a** ***math*** **test!!!** Use my square bracket notation and don't simplify your answers.

p131 #12

1. Use the definition of derivative to differentiate $y = \frac{1}{\sqrt{3x-2}}$. (No credit will be given unless you **use the definition of derivative!!!**)

Well, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\left(\frac{\frac{1}{\sqrt{3(x+h)-2}} - \frac{1}{\sqrt{3x-2}}}{h}\right)$

$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{3x-2} - \sqrt{3(x+h)-2}}{\sqrt{3x-2}\sqrt{3(x+h)-2}}\right)\left(\frac{\sqrt{3x-2}+\sqrt{3(x+h)-2}}{\sqrt{3x-2}+\sqrt{3(x+h)-2}}\right)$

$= \lim_{h\to 0} \frac{(3x-2)-(3x+3h-2)}{h\sqrt{3x-2}\sqrt{3(x+h)-2}\left(\sqrt{3x-2}+\sqrt{3(x+h)-2}\right)}$

$= \lim_{h\to 0} \frac{-3}{\sqrt{3x-2}\sqrt{3(x+h)-2}\left(\sqrt{3x-2}+\sqrt{3(x+h)-2}\right)}$

$= \frac{-3}{2(3x-2)^{3/2}}$

$\boxed{\frac{-3}{2(3x-2)^{3/2}}}$

p144 #59

2. The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and is tangent to the line $y = x$ at the origin. Find a, b, and c.

Well, $(x,y) = (1,2) \Rightarrow 2 = a(1)^2 + b(1) + c = a+b+c$. Since the curve passes through $(x,y) = (0,0)$, then $0 = a(0)^2 + b(0) + c \Rightarrow c = 0$.

Next, for $y = x$, $y' = 1$ and for $y = ax^2+bx+c$, $y' = 2ax+b$.

So at $(0,0)$, $1 = 2a(0) + b \Rightarrow b = 1$. Since $b = 1$, $c = 0$ and $2 = a+b+c$, then $a = 1$.

$\boxed{a=1,\ b=1,\ c=0}$

p219 #14a

3. Show that if the position x of a moving point is given by a quadratic function of t, $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1, t_2]$ is equal to the instantaneous velocity at the midpoint of the time interval.

Well, $$\left(\text{avg velocity}\right) = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{At_2^2 + Bt_2 + C - (At_1^2 + Bt_1 + C)}{t_2 - t_1}$$

$$= \frac{A(t_2^2 - t_1^2) + B(t_1 - t_2)}{t_2 - t_1} = A(t_2 + t_1) + B \quad \text{if } t_1 \neq t_2$$

and $$\left(\begin{array}{c}\text{inst. velocity} \\ \text{at } \frac{t_1 + t_2}{2}\end{array}\right) = x'\left(\frac{t_1 + t_2}{2}\right) = 2A\left(\frac{t_1 + t_2}{2}\right) + B$$

$$= A(t_2 + t_1) + B. \checkmark$$

p161 #60b

4. Use the facts that $\frac{d}{dx}[\sin x] = \cos x$ and $\frac{d}{dx}[\cos x] = -\sin x$, along with the Quotient Rule, to PROVE that the derivative of $\csc x$ is $-\csc x \cot x$.

Well, $$\frac{d}{dx}[\csc(x)] = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right] = \frac{[0](\sin x) - (1)[\cos(x)]}{(\sin x)^2}$$

$$= -\frac{1}{\sin x}\,\frac{\cos x}{\sin x} = -\csc x \cot x. \checkmark$$

HONORS!

5. If $y = \csc\left(\dfrac{e^{x^2} + 5x}{\ln(x^3 + \tan x)}\right)$ then $y' = ?$

$$y' = -\csc\left(\frac{e^{x^2} + 5x}{\ln(x^3 + \tan x)}\right)\cot\left(\frac{e^{x^2} + 5x}{\ln(x^3 + \tan x)}\right)\left[\frac{[e^{x^2}[2x] + 5](\ln(x^3 + \tan x)) - (e^{x^2} + 5x)\left[\frac{1}{x^3 + \tan(x)}[3x^2 + \sec^2(x)]\right]}{(\ln(x^3 + \tan x))^2}\right]$$

p 174 #27

6. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2,2)$.

Well, $\frac{d}{dx}[x^3+y^3] = \frac{d}{dx}[16] \Rightarrow 3x^2 + 3y^2\left[\frac{dy}{dx}\right] = 0$ and

$$\frac{dy}{dx} = \frac{-x^2}{y^2}. \text{ So } \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{-x^2}{y^2}\right] = \frac{[-2x](y^2) - (-x^2)\left[2y\left[\frac{dy}{dx}\right]\right]}{(y^2)^2}$$

$$= \frac{-2xy^2 + 2x^2y\frac{dy}{dx}}{y^4} = \frac{-2xy^2 + 2x^2y\left(\frac{-x^2}{y^2}\right)}{y^4} = \frac{-2xy^2 - \frac{2x^4}{y}}{y^4}.$$

When $(x,y) = (2,2)$, we have

$$\left.\frac{d^2y}{dx^2}\right|_{(2,2)} = \frac{-2(2)(2)^2 - \frac{2(2)^4}{(2)}}{(2)^4} = \frac{-16-16}{16} = \boxed{-2}$$

p185 #89

7. Find the derivative of $y = (x+1)^x$.

Well, we have $\ln y = \ln(x+1)^x = x\ln(x+1)$, and

so $\frac{d}{dx}[\ln y] = \frac{d}{dx}[x\ln(x+1)]$ or

$$\frac{1}{y}[y'] = [1](\ln(x+1)) + (x)\left[\frac{1}{x+1}\right] \text{ and}$$

$$y' = y\left(\ln(x+1) + \frac{x}{x+1}\right) = \boxed{(x+1)^x\left(\ln(x+1) + \frac{x}{x+1}\right)}$$

HONORS!

8. If $y = \sec^{-1}(\sin^{-1}(\tan^{-1}((\ln x)^2)))$ then $\frac{dy}{dx} = ?$

Well, by a bunch of applications of the Chain Rule,

$$\frac{dy}{dx} = \frac{1}{|\sin^{-1}(\tan^{-1}((\ln x)^2))|\sqrt{(\sin^{-1}(\tan^{-1}((\ln x)^2)))^2 - 1}}$$

$$\times\left[\frac{1}{\sqrt{1 - (\tan^{-1}((\ln x)^2))^2}} \cdot \frac{1}{1 + ((\ln x)^2)^2}\left[2(\ln x)\left[\frac{1}{x}\right]\right]\right]$$

p131 **Bonus 1.** Prove that if f has a derivative at $x = c$, then f is continuous at $x = c$.

Well, we need to show that $\lim_{x \to c} f(x) = f(c)$, or equivalently that $\lim_{h \to 0} f(c+h) = f(c)$:

$$\lim_{h \to 0} f(c+h) = \lim_{h \to 0} \left(f(c) + \frac{f(c+h) - f(c)}{h} \cdot h \right)$$

$$= \lim_{h \to 0} (f(c)) + \lim_{h \to 0} \left(\frac{f(c+h) - f(c)}{h} \right) \cdot \lim_{h \to 0} (h)$$

$$= f(c) + f'(c) \cdot (0) = f(c). \quad \blacksquare$$

p169 #100 **Bonus 2.** Suppose that the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s m from its starting point. Show that the body's acceleration is constant.

Well, acceleration $= \frac{d}{dt}[v] = \frac{d}{dt}[k \cdot s^{1/2}] = k \cdot \frac{1}{2} s^{-1/2} \left[\frac{ds}{dt}\right]$

$$= \frac{k}{2} \frac{1}{\sqrt{s}} \cdot v = \frac{k}{2} \frac{1}{\sqrt{s}} \cdot (k\sqrt{s}) = \frac{k^2}{2}$$

and $\frac{k^2}{2}$ is constant.