Honors Calculus 1 Test 3 — Fall 2010

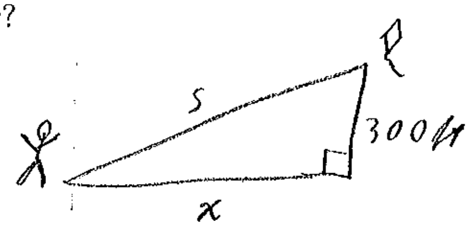
NAME KEY STUDENT NUMBER ____________________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols (such as limits and "element of" signs). If applicable, put your answer in the box provided. Each numbered problem is worth 12 points. **Put your calculators away!!! This is a *math* test!!!**

1, 2. A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

p 199 #25

1



2 Let s = amount of string, x = horizontal distance. We know $\frac{dx}{dt} = 25$ ft/sec.

3 The question is $\frac{ds}{dt} = ?$ when $s = 500$ ft

4 We know $x^2 + (300)^2 = s^2$

5 $\frac{d}{dt}[x^2 + 300^2] = \frac{d}{dt}[s^2]$,

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$

$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$$

6 When $s = 500$ ft, $x = 400$ ft and

$$\frac{ds}{dt} = \frac{(400\text{ ft})}{(500\text{ ft})}(25\text{ ft/sec}) = 20\text{ ft/sec}.$$

$\boxed{20 \text{ ft/sec}}$

3, 4. Consider $f(x) = x^{2/3}(x-5)$. Find the critical points, points of inflection, local MAXes and local MINs, intervals on which f in INC/DEC, intervals on which f is CU/CD, and graph $y = f(x)$.

$\boxed{\begin{array}{c} p\,252 \\ \#36 \end{array}}$

Well, $f'(x) = \left[\frac{2}{3}x^{-1/3}\right](x-5) + \left(x^{2/3}\right)[1] = \dfrac{2(x-5)}{3\sqrt[3]{x}} + x^{2/3}$

$$= \frac{2(x-5)+3x}{3\sqrt[3]{x}} = \frac{5x-10}{3\sqrt[3]{x}}.$$

So f has critical points at $x = 0$ (where f' is undefined) and $x = 2$ (where f' is 0).

Next, $f''(x) = \dfrac{[5](3\sqrt[3]{x}) - (5x-10)\left[x^{-2/3}\right]}{(3\sqrt[3]{x})^2}$

$$= \frac{15\sqrt[3]{x} - 5\sqrt[3]{x} + 10x^{-2/3}}{9x^{2/3}}$$

$$= \frac{10x+10}{9x^{4/3}}.$$

So $x = -1$ and $x = 0$ are potential inflection points ("PIPs")

Consider

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
k	-1	1	3
$f'(k)$	$\frac{-15}{-3}$	$\frac{-5}{3}$	$\frac{5}{3\sqrt[3]{3}}$
$f'(x)$	$+$	$-$	$+$
f	INC	DEC	INC

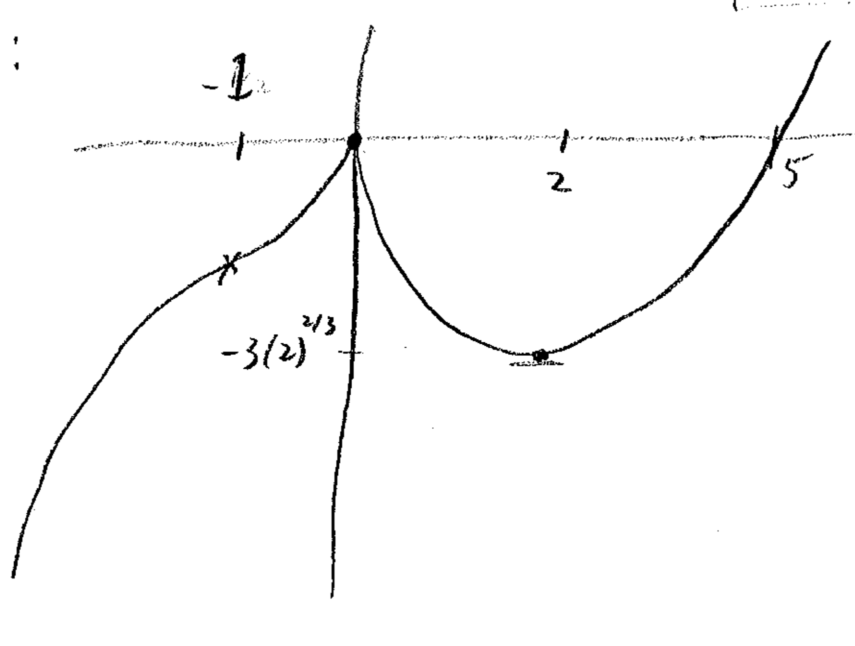
Consider

	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
k	-2	$-\frac{1}{2}$	1
$f''(k)$	$\frac{-10}{9(2)^{4/3}}$	$5/9(-\frac{1}{2})^{4/3}$	$\frac{30}{9}$
$f''(x)$	$-$	$+$	$+$
f	CD	CU	CU

So, $x = -1$ is a point of inflection.

$f(0) = 0$ is a local MAX
$f(2) = -3(2)^{2/3}$ is a local MIN

SO:



5. (a) Evaluate $\lim_{t\to 0} \frac{\sin 5t}{2t}$.

$\boxed{\begin{array}{l} p261 \\ \#14 \end{array}}$

$$\lim_{t\to 0} \frac{\sin(5t)}{(2t)} \overset{0/0}{=} \lim_{t\to 0}\left(\frac{5\cos(5t)}{2}\right) = \frac{5}{2}\cos(0) = \frac{5}{2}$$

$$\boxed{5/2}$$

$\boxed{\begin{array}{l} p261 \\ \#56 \end{array}}$ (b) Evaluate $\lim_{x\to\infty} x^{1/\ln x}$.

Let $y = x^{1/\ln(x)}$. Then $\ln(y) = \ln\left(x^{1/\ln(x)}\right) = \frac{\ln(x)}{\ln(x)} = 1.$

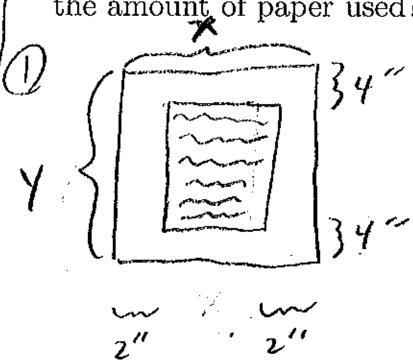
So $\lim_{x\to\infty}(\ln y) = \lim_{x\to\infty}(1) = 1$ and so $\lim_{x\to\infty}(y) = e^{\lim_{x\to\infty}(\ln y)} = e^1$

$$\boxed{e^1.}$$

6, 7. You are designing a rectangular poster to contain 50 in^2 of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used?

$\boxed{\begin{array}{l} p269 \\ \#11 \end{array}}$

①



② MIN Paper $= P = xy$
where x = width, y = height.

③ We know

$$\begin{pmatrix}\text{area of} \\ \text{print}\end{pmatrix} = (x-4)(y-8) = 50$$

$$\Rightarrow \quad y = \frac{50}{x-4} + 8$$

④ $$P = xy = \frac{50x}{x-4} + 8x$$

⑤ MIN $P(x) = \frac{50x}{x-4} + 8x$ for $x > 4$.

Well, $P'(x) = \frac{[50](x-4) - (50x)[1]}{(x-4)^2} + 8$

$$= \frac{-200}{(x-4)^2} + 8 = \frac{-200 + 8(x-4)^2}{(x-4)^2}$$

$$= \frac{-200 + 8x^2 - 64x + 128}{(x-4)^2} = \frac{8(x^2-8x-9)}{(x-4)^2}$$

$$= \frac{8(x-9)(x+1)}{(x-4)^2}.$$ So P has a critical value of $x = 9$.

Now, $P''(x) = \frac{400}{(x-4)^3}$ and $P''(9) = \frac{400}{(5)^3} > 0$

and so P has a MIN at $x = 9$.

$\boxed{\text{So, MIN when } x = 9 \text{ in and } y = 18 \text{ in.}}$

8. (a) Explain the difference between an antiderivative of f and the indefinite integral of f.

An antiderivative of f is a function F(x) such that $F'(x) = f(x)$ for all x in the domain of f. The indefinite integral of f is the set of all antiderivatives of f.

p287 #102

(b) Solve the initial value problem: $\frac{dv}{dt} = 8t + \csc^2 t$, and $v\left(\frac{\pi}{2}\right) = -7$.

$v(t) \in \int \left(\frac{dv}{dt}\right) dt = \int (8t + \csc^2(t))\,dt = 4t^2 - \cot(t) + C$.

So $v(t) = 4t^2 - \cot(t) + k$ for some k. Next, $v\left(\frac{\pi}{2}\right) = 4\left(\frac{\pi}{2}\right)^2 - \cot\left(\frac{\pi}{2}\right) + k$

$= -7$ or $\pi^2 - 0 + k = -7$ or $k = -7 - \pi^2$; so,

$$v(t) = 4t^2 - \cot(t) - 7 - \pi^2$$

$$\boxed{4t^2 - \cot(t) - 7 - \pi^2}$$

BONUS 1. Find the absolute maximum and minimum values of $f(x) = \sqrt{4 - x^2}$ on the interval $[-2, 1]$.

p228 #29

Notice $f(x) = (4 - x^2)^{1/2}$ and $f'(x) = \frac{1}{2}(4 - x^2)^{-1/2}[-2x]$

$= \frac{-x}{\sqrt{4 - x^2}}$. So $x = 0$ (where f' is 0), $x = -2$ and $x = 2$ (where f' is undefined) are critical points.

Consider

x	$f(x)$
-2	0
0	2
1	$\sqrt{3}$

MAX = 2
MIN = 2

So f has a MAX of 2 at $x = 0$
f has a MIN of 0 at $x = -2$

BONUS 2. (a) State the Mean Value Theorem.

$\boxed{p.231}$ Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point $c \in (a, b)$ at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$\boxed{p\,236 \ \#2}$ (b) Verify the hypotheses and conclusion of the Mean Value Theorem for $f(x) = x^{2/3}$ on $[0, 1]$.

Well, $f(x) = x^{2/3}$ is continuous on $[0,1]$ (in fact, it is continuous everywhere). Also, $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

and so f is differentiable on $(0, 1)$. Next,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow \frac{2}{3\sqrt[3]{c}} = \frac{(1)^{2/3} - (0)^{2/3}}{(1) - (0)}$$

$$\Rightarrow \frac{2}{3\sqrt[3]{c}} = 1 \Rightarrow \sqrt[3]{c} = \frac{2}{3} \text{ and } \boxed{c = \frac{8}{27}}.$$