Honors Calculus 1 Test 4 — Fall 2010

NAME KEY STUDENT NUMBER ____________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols (such as set notation and differentials). If applicable, put your answer in the box provided. Each numbered problem is worth 12 points. **Put your calculators away!!! This is a *math* test!!!**

1. State the definition of *Riemann integral* of function f on interval $[a, b]$, $\int_a^b f(x)\,dx$. Include the definition of *partition*, *norm* of a partition, and any other necessary parts.

pp. 311, 312, 314

A partition of $[a,b]$ is a set $P = \{x_0, x_1, \ldots, x_n\}$ where $a = x_0 < x_1 < \cdots < x_n = b$. P determines n subintervals $[x_0, x_1], \ldots, [x_{n-1}, x_n]$ and define $\Delta x_k = x_k - x_{k-1}$. The norm of P is $\|P\| = \max\{\Delta x_k\}$. A Riemann sum is $S_n = \sum_{k=1}^{n} f(c_k)\Delta x_k$ where $c_k \in [x_{k-1}, x_k]$. The Riemann integral is $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\left(\sum_{k=1}^{n} f(c_k)\Delta x_k\right)$.

2. Use a regular partition of $[0, 3]$, the right-hand endpoint of each subinterval for c_k, and **the definition of definite integral** (from Number 1) to evaluate $\int_0^3 2x\,dx$. Notice the formulae on the whiteboard.

p 313 #40

Well, with a regular partition $\Delta x_k = \Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$. So $x_k = x_0 + k\Delta x = 0 + k\left(\frac{3}{n}\right) = \frac{3k}{n}$. We have $\|P\| = \frac{3}{n}$ and so if $n \to \infty$ then $\|P\| \to 0$. Also, $c_k = x_k = \frac{3k}{n}$. So,

$$\int_a^b f(x)\,dx = \int_0^3 (2x)\,dx = \lim_{\|P\|\to 0}\left(\sum_{k=1}^{n} f(c_k)\Delta x_k\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f\left(\frac{3k}{n}\right)\left(\frac{3}{n}\right)\right)$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n} 2\left(\frac{3k}{n}\right)\left(\frac{3}{n}\right)\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\frac{18k}{n^2}\right)\right) = \lim_{n\to\infty}\left(\frac{18}{n^2}\sum_{k=1}^{n} k\right)$$

$$= \lim_{n\to\infty}\left(\frac{18}{n^2}\left(\frac{n(n+1)}{2}\right)\right) = 9\lim_{n\to\infty}\frac{n(n+1)}{n^2} = \boxed{9}.$$

3. State both parts of the Fundamental Theorem of Calculus (with hypotheses).

pp 327, 328

① If f is continuous on $[a,b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b) and

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

② If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

4. Evaluate $\int_0^\pi \frac{1}{2}(\cos x + |\cos x|)\,dx$.

p 333 #28

Well, $\int_0^\pi \frac{1}{2}(\cos x + |\cos x|)\,dx = \int_0^{\pi/2} \frac{1}{2}(\cos x + \cos x)\,dx$

$+ \int_{\pi/2}^{\pi} \frac{1}{2}(\cos(x) - \cos(x))\,dx$ (since $|\cos x| = \cos x$ for $x \in [0, \frac{\pi}{2}]$

and $|\cos x| = -\cos x$ for $x \in [\frac{\pi}{2}, \pi]$) $= \int_0^{\pi/2} \cos x\,dx = \sin(x)\Big|_0^{\pi/2}$

$= \sin(\frac{\pi}{2}) - \sin(0) = 1$

$\boxed{1}$

5. Evaluate $\int \frac{1}{\theta^2} \sin\frac{1}{\theta} \cos\frac{1}{\theta}\,d\theta$.

p 343 #35

let $u = \sin\left(\frac{1}{\theta}\right)$

$du = \cos\left(\frac{1}{\theta}\right)\left[\frac{-1}{\theta^2}\right] d\theta$

$-du = \frac{1}{\theta^2} \cos\left(\frac{1}{\theta}\right) d\theta$

$= \int u(-du) = -\int u\,du = -\frac{1}{2}u^2 + C$

$= -\frac{1}{2}\sin^2\left(\frac{1}{\theta}\right) + C$

$\boxed{-\frac{1}{2}\sin^2\left(\frac{1}{\theta}\right) + C}$

6. Evaluate $\int \frac{e^{\sin^{-1} x}\, dx}{\sqrt{1-x^2}}$

$\boxed{\begin{array}{l} p\,343 \\ \#61 \end{array}}$

$$\left[\begin{array}{l} \text{let } u = \sin^{-1}(x) \\ du = \frac{1}{\sqrt{1-x^2}}\,dx \end{array}\right] = \int e^u\, du = e^u + C$$

$$= e^{\sin^{-1}(x)} + C$$

$$\boxed{e^{\sin^{-1}(x)} + C}$$

7. Evaluate $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\, dx}{1 + (\cot x)^2}$.

$\boxed{\begin{array}{l} p\,350 \\ \#38 \end{array}}$

$$\left[\begin{array}{l} \text{let } u = \cot(x) \\ du = -\csc^2(x)dx \\ -du = \csc^2(x)dx \end{array}\right] = \int_{x=\pi/6}^{x=\pi/4} \frac{1}{1+u^2}(-du)$$

$$= -\tan^{-1}(u)\Big|_{x=\pi/6}^{x=\pi/4} = -\tan^{-1}\left(\cot\left(\tfrac{\pi}{4}\right)\right) + \tan^{-1}\left(\cot\left(\tfrac{\pi}{6}\right)\right)$$

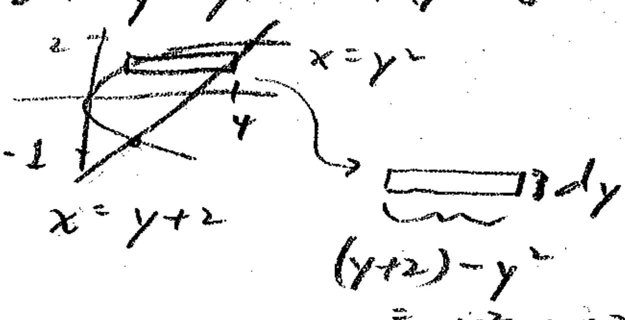
$$= -\tan^{-1}(1) + \tan^{-1}(\sqrt{3}) = -\frac{\pi}{4} + \frac{\pi}{3} = \frac{\pi}{12}$$

$$\boxed{\frac{\pi}{12}}$$

8. Find the area of the region bounded by $x = y^2$ and $x = y + 2$.

$\boxed{\begin{array}{l} p\,352 \\ \#74 \end{array}}$

Well, these intersect when $y^2 = y+2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0$
$\Rightarrow y = -1,\ y = 2$. The region is



The "area" of a dy slice is:
area $= (-y^2 + y + 2)dy$.
The total area is

$$A = \int_{-1}^{2} (-y^2 + y + 2)\,dy = \left(\frac{-1}{3}y^3 + \frac{1}{2}y + 2y\right)\Big|_{-1}^{2}$$

$$\boxed{\frac{9}{2}}$$

$$= \left(-\frac{8}{3} + 1 + 4\right) - \left(+\frac{1}{3} - \frac{1}{2} - 2\right) = \frac{-8}{3} + 5 - \frac{1}{3} + \frac{1}{2} + 2 = 4 + \frac{1}{2} = \boxed{\frac{9}{2}}$$

BONUS 1. If the average value of a function, av(f), really is a typical value of the integrable function $f(x)$ on $[a,b]$, then the constant function av(f) should have the same integral over $[a,b]$ as f. Does it? That is, does $\int_a^b \mathrm{av}(f),dx = \int_a^b f(x)\,dx$? Give reasons for your answer.

p323 #81

YES! $\int_a^b \mathrm{av}(f)\,dx = \mathrm{av}(f)\int_a^b dx = \mathrm{av}(f)(b-a)$.

Recall that $\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$, so

$$\int_a^b \mathrm{av}(f)\,dx = \mathrm{av}(f)(b-a) = \left(\frac{1}{(b-a)}\int_a^b f(x)\,dx\right)(b-a)$$

$$= \int_a^b f(x)\,dx \checkmark$$

BONUS 2. Find the linearization of $f(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt$ at $x=-1$. HINT: The linearization of f at $x=a$ is $L(x) = f(a) + f'(a)(x-a)$.

p335 #80

Well, $f(a) = f(-1) = 3 + \int_1^{(-1)^2} \sec(t-1)\,dt = 3 + 0 = 3$.

By the F.T.C. 1, $f'(x) = \frac{d}{dx}\left[3 + \int_1^{x^2} \sec(t-1)\,dt\right]$

Let $u = x^2$ and $f'(x) = \frac{d}{du}\left[3 + \int_1^{u} \sec(t-1)\,dt\right]\cdot\left[\frac{du}{dx}\right]$

$= \sec(u-1)\cdot\left[\frac{d[x^2]}{dx}\right] = \sec((x^2)-1)\cdot[2x] = 2x\sec(x^2-1)$,

and so $f'(-1) = 2(-1)\sec((-1)^2-1)$

$= -2\sec(0) = -2$.

$$\boxed{-2x+1}$$

So, $L(x) = 3 + -2(x-(-1)) = 3 - 2(x+1) = -2x+1$