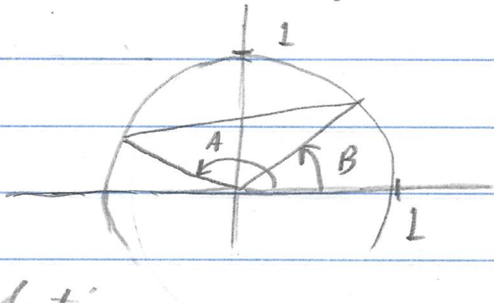
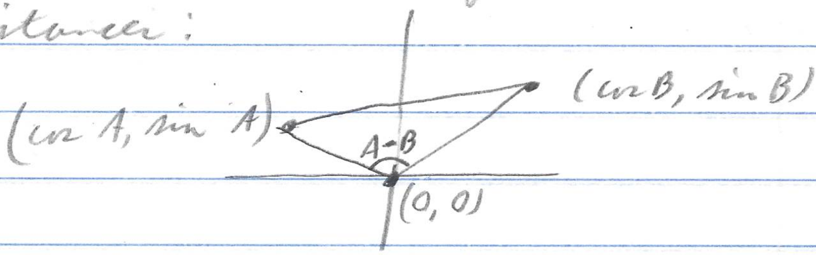
1.3 #57 Apply the Law of Cosines to the triangle below to derive the formula for $\cos(A-B)$:



Solution

We use the coordinates of the vertices to find distances:



(notice we are on a unit circle with $r = 1$ so that $\cos A = x/r = x$ and $\sin y = y/r = y$, for example). The length of the side opposite angle $A-B$ is $\sqrt{(\cos B - \cos A)^2 + (\sin B - \sin A)^2}$; the lengths of the other two sides are 1. By the Law of Cosines as applied to the triangle we have

$$\left(\sqrt{(\cos B - \cos A)^2 + (\sin B - \sin A)^2}\right)^2 = (1)^2 + (1)^2 - 2(1)(1)\cos(A-B)$$

or $(\cos B - \cos A)^2 + (\sin B - \sin A)^2 = 2 - 2\cos(A-B)$

or $\cos^2 B - 2\cos B \cos A + \cos^2 A + \sin^2 B - 2\sin B \sin A + \sin^2 A = 2 - 2\cos(A-B)$

or $(\cos^2 B + \sin^2 B) + (\cos^2 A + \sin^2 A) - 2(\cos A \cos B + \sin A \sin B) = 2 - 2\cos(A-B)$

or $(1)+(1) - 2(\cos A \cos B + \sin A \sin B)$
$= 2 - 2\cos(A-B)$

or $-2(\cos A \cos B + \sin A \sin B) = -2\cos(A-B)$,

or $\boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B}$. $\square$