1.6 #33 Consider $f(x) = x^2 - 2x$, $x \leq 1$. Find $f^{-1}(x)$, the domain and range of $f^{-1}(x)$ and check that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Solution

Well, let $y = f(x) = x^2 - 2x$, $x \leq 1$.
We interchange x and y to get
$x = y^2 - 2y$, $y \leq 1$ and then solve for y (giving $y = f^{-1}(x)$):

$$x = y^2 - 2y, \quad y \leq 1$$

or $$y^2 - 2y - x = 0, \quad y \leq 1$$

(Let's use the quadratic formula to solve for y; we have $a = 1$, $b = -2$, $c = -x$.)

or $$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-x)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 + 4x}}{2} = \frac{2 \pm \sqrt{4(1+x)}}{2}$$

$$\frac{2 \pm 2\sqrt{1+x}}{2} = 1 \pm \sqrt{1+x}, \quad y \leq 1.$$

Since $y \leq 1$ then we have $y = 1 - \sqrt{1+x}$.

So $\boxed{y = f^{-1}(x) = 1 - \sqrt{1+x}}$

With $f^{-1}(x) = 1 - \sqrt{1+x}$ we have the domain: $1 + x \geq 0$ (we can't take square roots of negatives) or $x \geq -1$. That is, the $\boxed{\text{domain is } [-1, \infty)}$

Recall: the range of $f^{-1}(x)$ is the domain of $f(x) = x^2 - 2x$, $x \leq 1$; the domain of f is $x \leq 1$, so the range of f^{-1} is $y \leq 1$. That is, the $\boxed{\text{range is } (-\infty, 1]}$.

Finally, we have

$$f(f^{-1}(x)) = f(1 - \sqrt{1+x})$$

$$= (1 - \sqrt{1+x})^2 - 2(1 - \sqrt{1+x})$$

$$= 1 - \cancel{2\sqrt{1+x}} + (\sqrt{1+x})^2 - 2 + \cancel{2\sqrt{1+x}}$$

$$= -1 + (1+x) = x. \checkmark$$

Also, $f^{-1}(f(x)) = f^{-1}(x^2 - 2x)$ where $x \leq 1$

$$= 1 - \sqrt{1 + (x^2 - 2x)} \text{ where } x \leq 1$$

$$= 1 - \sqrt{(x-1)^2} = 1 - |x-1| \text{ where } x \leq 1$$

$$= 1 - (-(x-1)) \text{ since } x - 1 \leq 0 \text{ and } |x-1| = -(x-1)$$

$$= x. \checkmark \quad \square$$