2.1.11 Find the slope of $y = x^3$ at the point $P(2, 8)$. Write an equation for the tangent line at this point. HINT: Choose a second point $Q(2+h, (2+h)^3)$ on the curve (where $h \neq 0$) and compute the slope of the secant line PQ. "Guess" what happens to the slope of the secant line when h is close to 0.

Solution

With the points $P(2,8)$ and $Q(2+h, (2+h)^3)$, the slope of the secant line to the curve through points P and Q is (since $h \neq 0$):

$$m = \frac{(2+h)^3 - (8)}{(2+h) - (2)} = \frac{(2+h)(4+4h+h^2) - 8}{2+h-2}$$

$$= \frac{8 + 8h + 2h^2 + 4h + 4h^2 + h^3 - 8}{h} = \frac{h^3 + 6h^2 + 12h}{h}$$

$= h^2 + 6h + 12$. For h "close to" 0 we have point Q "close to" point P and the secant line is "close to" the tangent line to the curve (which has a slope "close to" $(0)^2 + 6(0) + 12 = 12$). So we guess that the slope of the tangent line at $P(2,8)$ is $\boxed{m = 12}$.

The tangent line to $y = x^3$ at point $P(2,8) = (x_1, y_1)$, by the point-slope formula for a line $y - y_1 = m(x - x_1)$, is then $y - (8) = 12(x-2)$ or $y - 8 = 12x - 24$ or $\boxed{y = 12x - 16}$. □