2.1.17 Find the slope of $y = \sqrt{x}$ at the point $P(4,2)$. Write an equation for the tangent line at this point. HINT: Choose a second point $Q(4+h, \sqrt{4+h})$ on the curve (where $h \neq 0$) and compute the slope of the secant line PQ. "Guess" what happens to the slope of the secant line when h is close to 0.

Solution

With the points $P(4,2)$ and $Q(4+h, \sqrt{4+h})$, the slope of the secant line to the curve through points P and Q is (since $h \neq 0$):

$$m = \frac{\sqrt{4+h} - 2}{(4+h) - (4)} = \frac{\sqrt{4+h} - 2}{h}\left(\frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}\right)$$

$$= \frac{\left(\sqrt{4+h}\right)^2 - (2)^2}{h\left(\sqrt{4+h} + 2\right)} = \frac{(4+h) - 4}{h\left(\sqrt{4+h} + 2\right)} = \frac{h}{h\left(\sqrt{4+h} + 2\right)}$$

$$= \frac{1}{\sqrt{4+h} + 2}.$$

For h "close to" 0 we have point Q "close to" point P and the secant line is "close to" the tangent line to the curve (which has a slope "close to" $\frac{1}{\sqrt{4+(0)} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$).

So we guess that the slope of the tangent line at $P(4,2)$ is $\boxed{m = 1/4}$.

The tangent line to $y = \sqrt{x}$ at the point $P(4,2) = (x_1, y_1)$, by the point-slope formula for a line $y - y_1 = m(x - x_1)$, is then $y - (2) = \frac{1}{4}(x - 4)$ or $y - 2 = \frac{1}{4}x - 1$ or $\boxed{y = \frac{1}{4}x + 1}$. $\square$