2.2.65 (a) It can be shown that

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

holds for all values of x in some open interval containing $c = 0$ (except at $c = 0$).
What is $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$?

Solution

Let $g(x) = 1 - \frac{x^2}{6}$, $f(x) = \frac{x \sin x}{2 - 2\cos x}$,
and $h(x) = 1$. Notice that

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \left(1 - \frac{x^2}{6}\right) = 1 - \frac{(0)^2}{6} = 1$$

since g is a polynomial function
(Theorem 2.2, "Limits of Polynomials"),

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} (1) = 1.$$

That is, $\lim_{x \to 0} g(x) = \lim_{x \to 0} h(x) = 1 = L$.

By the Sandwich Theorem (Theorem 2.4)
we have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = L = \boxed{1}$.

$\square$