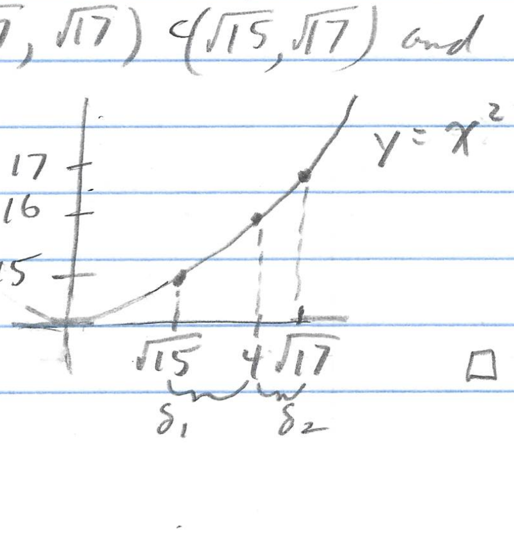
2.3.25 Consider $f(x) = x^2 - 5$, $L = 11$, $c = 4$, and $\varepsilon = 1$. Find an open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

<u>Solution</u>

First, notice that the domain of $f(x) = x^2 - 5$ is $x \in (-\infty, \infty) = \mathbb{R}$. The inequality $|f(x) - L| < \varepsilon$ or $|(x^2 - 5) - 11| < 1$ is equivalent to $-1 < x^2 - 16 < 1$ or $15 < x^2 < 17$ or $\sqrt{15} < \sqrt{x^2} < \sqrt{17}$ or $\sqrt{15} < |x| < \sqrt{17}$. Since we want x "close to $c = 4$" then we want x positive and so we consider $\sqrt{15} < x < \sqrt{17}$. So the desired inequality holds for $\boxed{x \in (\sqrt{15}, \sqrt{17})}$.

Now the distance from $c = 4$ to $\sqrt{15}$ is $\delta_1 = 4 - \sqrt{15}$ and the distance from $c = 4$ to $\sqrt{17}$ is $\delta_2 = \sqrt{17} - 4$. So we choose δ as the smaller of δ_1 and δ_2; that is, we take $\boxed{\delta = \sqrt{17} - 4}$. Then for $0 < |x - c| = |x - 4| < \delta = \sqrt{17} - 4$, we have $x \in (4 - \delta, 4 + \delta) = (8 - \sqrt{17}, \sqrt{17}) \subseteq (\sqrt{15}, \sqrt{17})$ and so the inequality holds.

NOTE: We see that $\delta_2 = \sqrt{17} - 4 < 4 - \sqrt{15} = \delta_2$ by consider the shape of the graph $y = x^2$:



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