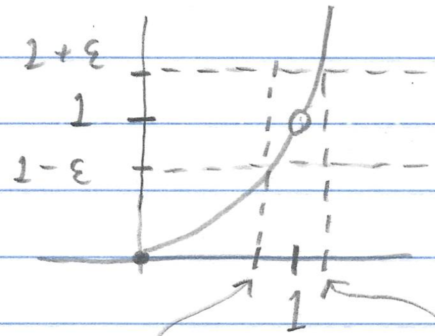
2.3.41 Prove $\lim_{x\to 1} f(x) = 1$ if $f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1. \end{cases}$

Proof

We have $c = 1$, $L = 1$, and $f(x)$ as given. Notice that f is defined on an open interval containing $c = 1$ (say the interval $(-\infty, \infty)$). Let $\varepsilon > 0$.

[THINKING (not part of proof): We want $|f(x) - L| < \varepsilon$ or $|f(x) - 1| < \varepsilon$ or $-\varepsilon < f(x) - 1 < \varepsilon$ or $1 - \varepsilon < f(x) < 1 + \varepsilon$. Consider the graph:



We need $1 - \varepsilon < x^2 < 1 + \varepsilon$ or $\sqrt{1-\varepsilon} < \sqrt{x^2} < \sqrt{1+\varepsilon}$ (since the square root function is increasing) or $\sqrt{1-\varepsilon} < |x| < \sqrt{1+\varepsilon}$.

Since x is "close to 1", then x is positive so we need $\sqrt{1-\varepsilon} < x < \sqrt{1+\varepsilon}$. So we take $\delta = \min\{1 - \sqrt{1-\varepsilon}, \sqrt{1+\varepsilon} - 1\} = \sqrt{1+\varepsilon} - 1$ (notice that the distance on the right $\sqrt{1+\varepsilon} - 1$ is smaller than the distance on the left $1 - \sqrt{1-\varepsilon}$, based on the shape of $y = x^2$.]

Let $\delta = \sqrt{1+\varepsilon} - 1$. If $0 < |x - c| < \delta$ then $0 < |x - 1| < \delta = \sqrt{1+\varepsilon} - 1$ which is equivalent to $-(\sqrt{1+\varepsilon} - 1) < x - 1 < \sqrt{1+\varepsilon} - 1$ and $x \neq 1$.

This gives $-\sqrt{1+\varepsilon}+1 < x-1 < \sqrt{1+\varepsilon}-1$ and $x \neq 1$
or, since $1-\sqrt{1-\varepsilon} \geq \sqrt{1+\varepsilon}-1$
(or equivalently $\sqrt{1-\varepsilon}-1 \leq 1-\sqrt{1+\varepsilon} = -\sqrt{1+\varepsilon}+1$)
as discussed above,

$$\sqrt{1-\varepsilon}-1 \leq -\sqrt{1+\varepsilon}+1 < x-1 < \sqrt{1+\varepsilon}-1$$

and $x \neq 1$, which implies

$$\sqrt{1-\varepsilon} < x < \sqrt{1+\varepsilon} \quad \text{or} \quad \left(\sqrt{1-\varepsilon}\right)^2 < x^2 < \left(\sqrt{1+\varepsilon}\right)^2$$

since the squaring function is increasing for nonnegative inputs, or $1-\varepsilon < x^2 < 1+\varepsilon$
or $-\varepsilon < x^2-1 < \varepsilon$ or $|x^2-1| < \varepsilon$

or $|f(x)-L| < \varepsilon$. Therefore, by the definition of limit, $\lim_{x \to c} f(x) = L$ or $\lim_{x \to 1} f(x) = 1$,

as claimed. ∎