2.3.49 Prove $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Proof

We have $f(x) = x \sin\left(\frac{1}{x}\right)$, $c = 0$, and $L = 0$. Notice that f is defined on an open interval containing $c = 0$ except at $c = 0$ (say the interval $(-\infty, \infty)$). Let $\varepsilon > 0$.

[THINKING (not part of proof): We want $|f(x) - L| < \varepsilon$ or $|x \sin(1/x) - 0| < \varepsilon$ or $|x \sin(1/x)| < \varepsilon$ or (since $|\sin(1/x)| \leq 1$) $|x| \leq |x \sin(1/x)| < \varepsilon$ which gives $|x - 0| < \varepsilon$. So we take $\delta = \varepsilon > 0$ and $0 < |x - c| < \delta$ or $0 < |x - 0| < \delta$ or $0 < |x| < \delta$ and we can reverse the above computations.]

Let $\delta = \varepsilon$. If $0 < |x - c| < \delta$ then $0 < |x - 0| < \delta = \varepsilon$ or $0 < |x| < \varepsilon$. Now $|\sin(1/x)| \leq 1$ for all $x \neq 0$, so $0 < |x| < \varepsilon$ implies
$$0 \leq |x|\,|\sin(1/x)| = |x \sin(1/x)| \leq |x| < \varepsilon,$$
which implies $|x \sin(1/x) - 0| < \varepsilon$ or $|f(x) - L| < \varepsilon$. Therefore, by the definition of limit, $\lim_{x \to c} f(x) = L$ or $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$, as claimed. ∎

NOTE From the figure in the book, we see $\left|x \sin\left(\frac{1}{x}\right)\right| \leq |x|$.