

2.4.29 Evaluate $\lim_{x \rightarrow 0} \frac{x \csc(2x)}{\cos(5x)}$. Explain.

Solution

We have

$$\lim_{x \rightarrow 0} \frac{x \csc(2x)}{\cos(5x)} = \lim_{x \rightarrow 0} \frac{x}{\cos(5x)} \frac{1}{\sin(2x)}$$

$$\text{since } \csc \theta = \frac{1}{\sin \theta}$$

$$= \lim_{x \rightarrow 0} \frac{1}{\cos(5x)} \frac{1}{2} \frac{(2x)}{\sin(2x)} \quad \text{since we want the same argument in sine as in the numerator}$$

$$= \lim_{\theta \rightarrow 0} \frac{1}{2 \cos(5(\theta/2))} \frac{2(\theta/2)}{\sin(2(\theta/2))} \quad \text{replacing } x \text{ with } \theta/2 \text{ and}$$

observing that $x \rightarrow 0$ if and only if $\theta \rightarrow 0$

$$= \frac{1}{2} \lim_{\theta \rightarrow 0} \frac{1}{\cos(5\theta/2)} \frac{1}{(\sin \theta)/\theta} \quad \text{by the Constant Multiple Rule (Theorem 1.1(3))}$$

$$= \frac{1}{2} \lim_{\theta \rightarrow 0} \frac{1}{\cos(5\theta/2)} \lim_{\theta \rightarrow 0} \frac{1}{(\sin \theta)/\theta}$$

by the Product Rule (Theorem 1.1(4))

$$= \frac{1}{2} \frac{1}{\lim_{\theta \rightarrow 0} \cos(5\theta/2)} \frac{1}{\lim_{\theta \rightarrow 0} (\sin \theta)/\theta} \quad \text{by the Quotient Rule (Theorem 1.1(5))}$$

$$= \frac{1}{2} \frac{1}{(1)} \frac{1}{(1)} \quad \text{by Example 2.2.11(b) and Theorem 2.7}$$

$$= \boxed{\frac{1}{2}}. \quad \square$$