

2.4.41 Evaluate $\lim_{x \rightarrow 0} \frac{\tan(3x)}{\sin(8x)}$. Explain.

Solution

We have

$$\lim_{x \rightarrow 0} \frac{\tan(3x)}{\sin(8x)} = \lim_{x \rightarrow 0} \frac{\sin(3x)}{\cos(3x)} \frac{1}{\sin(8x)}$$

$$= \lim_{x \rightarrow 0} \frac{3}{8} \frac{\sin(3x)}{3x} \frac{8x}{\sin(8x)} \frac{1}{\cos(3x)}$$

$$= \frac{3}{8} \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} \lim_{x \rightarrow 0} \frac{8x}{\sin(8x)} \lim_{x \rightarrow 0} \frac{1}{\cos(3x)}$$

by Theorem 1.2 (3,4), Constant Multiple Rule and Product Rule

$$= \frac{3}{8} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \lim_{\varphi \rightarrow 0} \frac{\varphi}{\sin \varphi} \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta}$$

letting $\theta = 3x$ and $\varphi = 8x$ and the facts that
 $x \rightarrow 0$ if and only if $\theta \rightarrow 0$, and
 $x \rightarrow 0$ if and only if $\varphi \rightarrow 0$

$$= \frac{3}{8} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \frac{1}{\lim_{\varphi \rightarrow 0} (\sin \varphi) / \varphi} \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta}$$

$$= \frac{3}{8} (1) \frac{1}{(1)} \frac{1}{(1)} \text{ by Theorem 2.1(5), Quotient Rule}$$

by Theorem 2.7 and Exercise 2.2.11(b)

$$= \boxed{\frac{3}{8}}. \quad \square$$