2.4.51 Given $\varepsilon > 0$, find an interval $I = (5, 5+\delta)$, $\delta > 0$, such that if $x \in I$ then $\sqrt{x-5} < \varepsilon$. What limit is being verified and what is its value?

Solution

If $\sqrt{x-5} < \varepsilon$ then $(\sqrt{x-5})^2 < (\varepsilon)^2$ (since the squaring function is increasing for positive inputs) or $x-5 < \varepsilon^2$ or $x < 5 + \varepsilon^2$. So we take $\boxed{\delta = \varepsilon^2}$ and for $x \in I \in (5, 5+\delta)$ we have $5 < x < 5+\delta = 5+\varepsilon^2$ which implies $0 < x-5 < \varepsilon^2$ or $\sqrt{x-5} < \sqrt{\varepsilon^2} = |\varepsilon| = \varepsilon$, as desired.

Here we have $c = 5$, $L = 0$, and $f(x) = \sqrt{x-5}$. Since we consider $x \in (5, 5+\delta)$ then the limit under consideration is

$$\boxed{\lim_{x \to c} f(x) = \lim_{x \to 5^+} \sqrt{x-5} \text{ which has value } 0.}$$

Notice that $\sqrt{x-5} < \varepsilon$ is then equivalent to the inequality $|\sqrt{x-5} - 0| < \varepsilon$ or $|f(x) - L| < \varepsilon$. $\square$