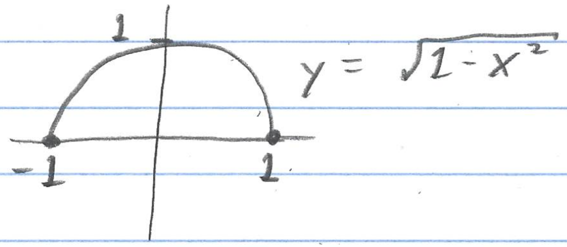
2.4.9 Consider $f(x) = \begin{cases} \sqrt{1-x^2}, & x \in [0,1) \\ 1, & x \in [1,2) \\ 2, & x = 2 \end{cases}$.

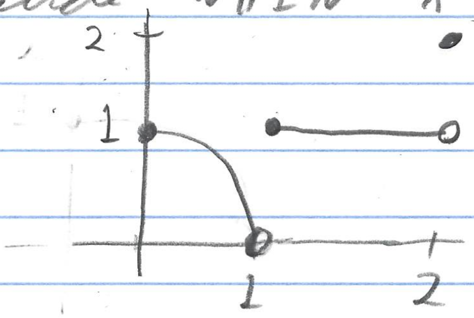
Graph it. Discuss limits.

<u>Solution</u>

Well $y = \sqrt{1-x^2}$ implies $y^2 = \left(\sqrt{1-x^2}\right)^2 = 1 - x^2$ or $x^2 + y^2 = 1$ AND $\underline{y \geq 0}$. So:



So the graph of $y = f(x)$ is this semicircle WHEN $x \in [0, 1)$, and hence



Now for limits. For two-sided limits, $\lim_{x \to 0} f(x)$ does not exist because f is not defined on an open interval containing 0. Similarly, $\lim_{x \to 2} f(x)$ does not exist.

What about $c = ?$ Well, the graph "tries to contain" the point $(1, 0)$ as $x \to 1^-$, so $\lim_{x \to 1^-} f(x) = 0$.

As $x \to 1^+$ the graph tries to contain the point $(1, 1)$ and so $\lim_{x \to 1^+} f(x) = 1$.

So $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$ and hence by Theorem 2.6, "Relation Between One and Two-Sided Limits", we have $\lim_{x \to 1} f(x)$ does not exist.

For all other values of c in $(0, 1) \cup (1, 2)$, the two-sided limit $\lim_{x \to c} f(x)$ exists by, say, Dr. Bob's anthropomorphic Definition of Limit. So the two-sided limit exists for at c for $c \in (0, 1) \cup (1, 2)$. (Notice that the answer in the back of the book erroneously includes 0 and 2.) For one-sided limits, $\lim_{x \to 0^-} f(x)$ does not exist since f is not defined for $x < 0$. Similarly, $\lim_{x \to 2^+} f(x)$ does not exist.

As $x \to 0^+$, the graph tries to contain the point $(0, 1)$ (in fact it does contain the point), so $\lim_{x \to 0^+} f(x) = 1$. As $x \to 1^-$, the graph tries to contain the point $(1, 0)$ (it does NOT contain the point), so $\lim_{x \to 1^-} f(x) = 0$. As $x \to 1^+$ the graph tries to contain the point $(1, 1)$ (it does) so $\lim_{x \to 1^+} f(x) = 1$. As $x \to 2^-$ the graph tries to contain the point $(2, 1)$ (it doesn't) so $\lim_{x \to 2^-} f(x) = 1$. Each of these one-sided limit values are justified by Dr. Bob's Anthropomorphic Definition of Limit applied to one-sided limits.

$\boxed{\text{So the left-hand limit exists at } c = 2 \text{ but the right-hand limit does not exist.}}$

$\boxed{\text{The right-hand limit exists at } x = 0 \text{ but the left-hand limit does not exist.}}$

From the graph of $y = f(x)$, notice the $\boxed{\text{domain is } [0, 2]}$ and the $\boxed{\text{range is } (0, 1) \cup \{2\}.}$ $\square$