2.5.33 Find $\lim_{x \to \pi} \sin(x - \sin x)$. Is the function continuous at the point $x = \pi$?

Solution

First, $\sin((\pi) - \sin(\pi)) = \sin(\pi - 0) = \sin(\pi) = 0$.
The limit involves compositions of functions! Since the sine function is continuous on its domain (by Theorem 2.5.A) then:

$$\lim_{x \to \pi} \sin(x - \sin x) = \sin\left(\lim_{x \to \pi}(x - \sin x)\right)$$

since sine is continuous and the definition of "continuous"

$$= \sin\left(\lim_{x \to \pi}(x) - \lim_{x \to \pi}(\sin(x))\right)$$

by Theorem 2.1.A (2), Difference Rule

$$= \sin\left(\lim_{x \to \pi}(x) - \sin\left(\lim_{x \to \pi}(x)\right)\right)$$

since sine is continuous

$$= \sin(\pi - \sin(\pi))$$ since x is a polynomial; Theorem 2.5.A

$$= \boxed{0}.$$

Notice that $\lim_{x \to \pi} \sin(x - \sin x) = 0$,

so with $f(x) = \sin(x - \sin x)$ we have

① $f(0)$ exists and $f(0) = 0$,

② $\lim_{x \to \pi} f(x) = 0$, and

③ $\lim_{x \to \pi} f(x) = f(\pi)$.

So, by the Test for Continuity, $\boxed{\text{YES}}$ this function is continuous at $x = \pi$.

□