2.5.69 <u>A Fixed Point Theorem</u>

Suppose that a function f is continuous on the closed interval $[0,1]$ and that $0 \leq f(x) \leq 1$ for every $x \in [0,1]$. Prove that there must exist a number $c \in [0,1]$ such that $f(c) = c$ (c is called a <u>fixed point</u> of f).

<u>Proof</u>

If $f(0) = 0$ then we can take $c = 0$. If $f(1) = 1$ then we can take $c = 1$. So without loss of generality we assume $f(0) > 0$ and $f(1) < 1$.

Consider $g(x) = f(x) - x$. Since we now have $f(0) > 0$ then $g(0) = f(0) - (0) = f(0) > 0$. Since we now have $f(1) < 1$ then $g(1) = f(1) - (1) < 1 - 1 = 0$. Since f is continuous on $[0,1]$ by hypothesis and the function x is continuous (it is a polynomial, say; by Theorem 2.5.A) then $g(x) = f(x) - x$ is continuous Theorem 2.8(2). So g is continuous on $[0,1]$, $g(0) > 0$, and $g(1) < 0$. Since $y_0 = 0$ is between $g(0)$ and $g(1)$ then by the Intermediate Value Theorem (Theorem 2.11), there is $c \in [0,1]$ such that $g(c) = f(c) - c = 0$. So for this value, $f(c) = c$ as claimed. ■