2.6.73 Consider $y = f(x) = \dfrac{\sqrt{x^2+4}}{x}$. Find the domain, range, limits, and asymptotes.

<u>Solution</u>

First, we cannot divide by 0, so 0 is not in the domain. (Notice that $x^2+4>0$, so square roots negatives aren't a problem here.) So, the $\boxed{\text{domain is } (-\infty, 0) \cup (0, \infty)}$.

Lets consider some infinite limits:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{\sqrt{x^2+4}}{x}\right) = \lim_{x\to\infty}\left(\frac{\sqrt{x^2+4}}{x}\right)\left(\frac{1/x}{1/x}\right)$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2+4}\,(1/\sqrt{x^2})}{x/x}$$ since $1/\sqrt{x^2} = 1/|x|$ $= 1/x$ for $x>0$ (hey, $x \to +\infty$)

$$= \lim_{x\to\infty}\left(\frac{\sqrt{\frac{x^2+4}{x^2}}}{x/x}\right) = \lim_{x\to\infty}\frac{\sqrt{1+\frac{4}{x^2}}}{1}$$

$$= \lim_{x\to\infty}\sqrt{1+\frac{4}{x^2}} = \sqrt{\lim_{x\to\infty}(1) + 4\left(\lim_{x\to\infty}\frac{1}{x}\right)^2}$$

$4\,\frac{1}{x^2}$

by Theorem 2.12 (7, 1, 4, 6), the Root Rule, Sum Rule, Constant Multiple Rule, and the Power Rule (respectively)

$$= \sqrt{1 + 4(0)} \quad \text{by Example 2.6.1}$$

$$= \sqrt{1+0} = 1.$$

So, $$\boxed{\lim_{x \to \infty} \left(\frac{\sqrt{x^2+4}}{x} \right) = 1.}$$

Next, $$\lim_{x \to -\infty} \frac{\sqrt{x^2+4}}{x} = \lim_{x \to -\infty} \left(\frac{\sqrt{x^2+4}}{x} \right) \left(\frac{1/x}{1/x} \right)$$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^2+4}\,(-1/\sqrt{x^2})}{x/x}$$ since $1/\sqrt{x^2} = 1/|x|$

$= -1/x$ for $x < 0$

(hey, $x \to -\infty$)

$$= \lim_{x \to -\infty} \frac{-\sqrt{1 + 4/x^2}}{x/x} = \lim_{x \to -\infty} \frac{-\sqrt{1+4/x^2}}{1}$$

$$= -\sqrt{\lim_{x \to -\infty} (1) + 4 \left(\lim_{x \to -\infty} \frac{1}{x} \right)^2}$$

by Theorem 2.12(4, 7, 1, 6)
Constant Multiple Rule, Root Rule, Sum Rule,
Power Rule (respectively)

$$= -\sqrt{1 + 4(0)} = -1.$$

So $$\boxed{\lim_{x \to -\infty} \left(\frac{\sqrt{x^2+4}}{x} \right) = -1.}$$

Next, $f(x)$ is a quotient of functions where the numerator has a limit as $x \to 0$ of $\sqrt{(0)^2+4} = 2$, and the denominator has a limit as $x \to 0$ of 0. So by Dr. Bob's Infinite Limit Theorem, $\lim_{x \to 0^\pm} \frac{\sqrt{x^2+4}}{x} = \pm\infty$.

We consider a "SIGN DIAGRAM" for $\frac{\sqrt{x^2+4}}{x}$.

As $x \to 0^-$ we have

$\frac{\sqrt{x^2+4}}{x} \Rightarrow \frac{(+)}{(-)} = -$ and as $x \to 0^+$ we have

$\frac{\sqrt{x^2+4}}{x} \Rightarrow \frac{(+)}{(+)} = +$. So

$$\boxed{\lim_{x \to 0^-} \frac{\sqrt{x^2+4}}{x} = -\infty \text{ and } \lim_{x \to 0^+} \frac{\sqrt{x^2+4}}{x} = +\infty.}$$

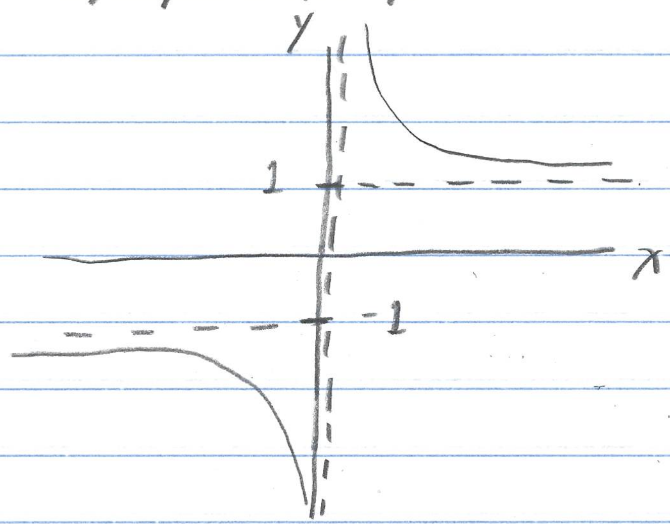
Since $\lim_{x \to \infty} f(x) = 1$ and $\lim_{x \to -\infty} f(x) = -1$, then

$$\boxed{y = 1 \text{ and } y = -1 \text{ are horizontal asymptotes.}}$$

Since $\lim_{x \to 0^-} f(x) = -\infty$ and $\lim_{x \to 0^+} f(x) = \infty$,

then $\boxed{x = 0 \text{ is a vertical asymptote.}}$

The graph of $y = f(x) = \dfrac{\sqrt{x^2+4}}{x}$ is



(Notice that $\left|\dfrac{\sqrt{x^2+4}}{x}\right| > \left|\dfrac{\sqrt{x^2}}{x}\right| = 1$,

so the function never takes on values between -1 and 1.)

$\boxed{\text{So the range is } (-\infty, -1) \cup (1, \infty).}$ □