2.6.85 How many horizontal asymptotes can the graph of a given rational function have?

Solution

Consider rational function $f(x) = \frac{p(x)}{q(x)}$.
First, suppose p and q are of the same degree, say n. Let the coefficients of p be $a_0, a_1, a_2, \ldots, a_n$ and let the coefficients of q be $b_0, b_1, b_2, \ldots, b_n$. Then:

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{p(x)}{q(x)}$$

$$= \lim_{x \to \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0}$$

$$= \lim_{x \to \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_n x^n + a_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0}\left(\frac{1/x^n}{1/x^n}\right)$$

$$= \lim_{x \to \pm\infty} \frac{a_n \frac{x^n}{x^n} + a_{n-1}\frac{x^{n-1}}{x^n} + \cdots + a_1 \frac{x}{x^n} + a_0 \frac{1}{x^n}}{b_n \frac{x^n}{x^n} + b_{n-1}\frac{x^{n-1}}{x^n} + \cdots + b_1 \frac{x}{x^n} + b_0 \frac{1}{x^n}}$$

$$= \lim_{x \to \pm\infty} \frac{a_n + a_{n-1}\left(\frac{1}{x}\right) + \cdots + a_1\left(\frac{1}{x}\right)^{n-1} + a_0\left(\frac{1}{x}\right)^n}{b_n + b_{n-1}\left(\frac{1}{x}\right) + \cdots + b_1\left(\frac{1}{x}\right)^{n-1} + b_0\left(\frac{1}{x}\right)^n}$$

$$= \frac{a_n + a_{n-1}\left(\lim_{x \to \pm\infty} \frac{1}{x}\right) + \cdots + a_1\left(\lim_{x \to \pm\infty} \frac{1}{x}\right)^{n-1} + a_0\left(\lim_{x \to \pm\infty} \frac{1}{x}\right)^n}{b_n + b_{n-1}\left(\lim_{x \to \pm\infty} \frac{1}{x}\right) + \cdots + b_1\left(\lim_{x \to \pm\infty} \frac{1}{x}\right)^{n-1} + b_0\left(\lim_{x \to \pm\infty} \frac{1}{x}\right)^n}$$

$$= \frac{a_n + a_{n-1}(0) + \cdots + a_1(0)^{n-1} + a_0(0)^n}{b_n + b_{n-1}(0) + \cdots + b_1(0)^{n-1} + a_0(0)^n} = \frac{a_n}{b_n}.$$

In this case, f has one horizontal asymptote.
If p is of degree $m < n$ then, as above,

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{p(x)}{q(x)}$$

$$= \lim_{x \to \pm\infty} \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0}$$

$$= \lim_{x \to \pm\infty} \frac{a_m\left(\frac{1}{x}\right)^{n-m} + a_{m-1}\left(\frac{1}{x}\right)^{n-m+1} + \cdots + a_1\left(\frac{1}{x}\right)^{n-1} + a_0\left(\frac{1}{x}\right)^n}{b_n + b_{n-1}\left(\frac{1}{x}\right) + \cdots + b_1\left(\frac{1}{x}\right)^{n-1} + b_0\left(\frac{1}{x}\right)^n}$$

dividing numerator and denominator by x^n, distributing, and simplifying

$$= \frac{a_m(0)^{n-1} + a_{m-1}(0)^{n-m+1} + \cdots + a_1(0)^{n-1} + a_0(0)^n}{b_n + b_{n-1}(0) + \cdots + b_1(0)^{n-1} + b_0(0)}$$

using Theorem 2.12 and Example 2.6.1, as above

$$= 0.$$

So in this case, f has one horizontal asymptote.
If p is of degree m greater than the degree n of q, then the dominant terms of the rational function f is itself a polynomial of degree $m-n$ and so f will not have a horizontal asymptote.
Since this covers all possible degrees of p and q, then we see that

$\boxed{\text{a rational function can have at most one horizontal asymptote.}}$ $\square$