2.6.89 Find $\lim_{x \to -\infty} \left(2x + \sqrt{4x^2+3x-2}\right)$. HINT: Try multiplying and dividing by the conjugate).

Solution

We have

$$\lim_{x \to -\infty} \left(2x+\sqrt{4x^2+3x-2}\right)$$

$$= \lim_{x \to -\infty} \left(2x+\sqrt{4x^2+3x-2}\right)\left(\frac{2x-\sqrt{4x^2+3x-2}}{2x-\sqrt{4x^2+3x-2}}\right)$$

from the hint

$$= \lim_{x \to -\infty} \frac{(2x)^2-\left(\sqrt{4x^2+3x-2}\right)^2}{2x-\sqrt{4x^2+3x-2}}$$

$$= \lim_{x \to -\infty} \frac{4x^2-(4x^2+3x-2)}{2x-\sqrt{4x^2+3x-2}}$$

$$= \lim_{x \to -\infty} \frac{-3x+2}{2x-\sqrt{4x^2+3x-2}}\left(\frac{1/x}{1/x}\right)$$

simplifying and dividing numerator and denominator by the effective highest power of x in the denominator

$$= \lim_{x \to -\infty} \frac{-3\left(\frac{x}{x}\right)+2\left(\frac{1}{x}\right)}{2\left(\frac{x}{x}\right)-\sqrt{4x^2+3x-2}\left(\frac{-1}{\sqrt{x^2}}\right)}$$

since $\frac{1}{\sqrt{x^2}} = \frac{1}{|x|} = \frac{-1}{x}$ since $x<0$ (hey, $x \to -\infty$)

$$= \lim_{x \to -\infty} \frac{-3 + 2\left(\frac{1}{x}\right)}{2 + \sqrt{4\left(\frac{x^2}{x^2}\right) + 3\left(\frac{x}{x^2}\right) - 2\left(\frac{1}{x^2}\right)}}$$

$$= \lim_{x \to -\infty} \frac{-3 + 2\left(\frac{1}{x}\right)}{2 + \sqrt{4 + 3\left(\frac{1}{x}\right) - 2\left(\frac{1}{x}\right)^2}}$$

$$= \frac{-3 + 2 \lim_{x \to -\infty}\left(\frac{1}{x}\right)}{2 + \sqrt{4 + 3\left(\lim_{x \to -\infty} \frac{1}{x}\right) - 2\left(\lim_{x \to -\infty} \frac{1}{x}\right)^2}}$$

by Theorem 2.12 (5, 1, 7, 2, 6)
Quotient Rule, Sum Rule, Root Rule,
Difference Rule, and Power Rule (respectively)

$$= \frac{-3 + 2(0)}{2 + \sqrt{4 + 3(0) - 2(0)^2}}$$ by Example 2.6.1

$$= \frac{-3}{2 + \sqrt{4}} = \boxed{\frac{-3}{4}}$$ □