3.1.25 At what points do the graph of $f(x) = x^2 + 4x - 1$ has horizontal tangent lines?

<u>Solution</u>

Well, the slope of a curve at $x = x_0$ is

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h},$$ the slope of a

horizontal line is 0, so the question is

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = 0 \Rightarrow x_0 = ?$$

Here

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{\left((x_0+h)^2 + 4(x_0+h) - 1\right) - \left(x_0^2 + 4x_0 - 1\right)}{h}$$

$$= \lim_{h \to 0} \frac{x_0^2 + 2hx_0 + h^2 + 4x_0 + 4h - 1 - x_0^2 - 4x_0 + 1}{h}$$

$$= \lim_{h \to 0} \frac{2hx_0 + h^2 + 4h}{h} \overset{F}{=} \lim_{h \to 0} \frac{h(2x_0 + h + 4)}{h}$$

$$\overset{C}{=} \lim_{h \to 0} (2x_0 + h + 4) \overset{S}{=} 2x_0 + (0) + 4 = 2x_0 + 4$$

since $2x_0 + h + 4$ is a polynomial in h; see Theorem 2.2.

So, we need $m = 2x_0 + 4 \equiv 0$ and hence $x_0 = -2$. At $(x_0, f(x_0)) = (-2, (-2)^2 + 4(-2) - 1)$ $= \boxed{(-2, -5)}$ there is a horizontal tangent. □