3.1.37 Does the graph of $f(x) = \begin{cases} -1, & x<0 \\ 0, & x=0 \\ 1, & x>0 \end{cases}$ have a vertical tangent line at the origin?

Solution

We address $\displaystyle\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ (since at the origin $x=0$) and because $f$ is piecewise defined, we consider one-sided limits:

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(-1)-(0)}{h} \text{ since } h<0$$

$$= \lim_{h\to 0^-} \frac{-1}{h} = \infty$$ by Dr. Bob's Infinite Limits Theorem and the sign diagram $\frac{-1}{h} \to \frac{(-)}{(-)} = +$, AND

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(1)-(0)}{h} \text{ since } h>0$$

$$= \lim_{h\to 0^+} \frac{1}{h} = \infty$$ by Dr. Bob's Infinite Limits Theorem and the sign diagram $\frac{1}{h} \to \frac{(+)}{(+)} = +$.

So we do have $\displaystyle\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \infty$,

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as needed for the existence of a vertical tangent line. However, from the definition of vertical tangent line at $x = x_0$ (given in the book before the statement of this problem), we need $f$ to be continuous at $x = x_0 = 0$ (here). $f$ has a jump discontinuity at $x_0 = 0$ so $\boxed{\text{NO}}$ $f$ does not have a vertical tangent at the origin. $\square$