3.6.9 Find an equation of the tangent line to $y = x^3$ at the point $(-2, -8)$. Graph.

Solution

Let $y = f(x) = x^3$ and let $(x_0, f(x_0)) = (-2, -8)$. The slope of the desired line is

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{f((-2)+h) - f(-2)}{h} \quad \text{since } x_0 = -2$$

$$= \lim_{h \to 0} \frac{(-2+h)^3 - (-2)^3}{h} \quad \text{since } f(x) = x^3$$

$$= \lim_{h \to 0} \frac{(-2+h)(4-4h+h^2) - (-8)}{h}$$

$$= \lim_{h \to 0} \frac{-8+8h-2h^2+4h-4h^2+h^3+8}{h}$$

$$= \lim_{h \to 0} \frac{12h-6h^2+h^3}{h} = \lim_{h \to 0} \frac{h(12-6h+h^2)}{h}$$

← Factor

"FCS"

← Cancelled

$$= \lim_{h \to 0} (12-6h+h^2) \quad \text{since } h \neq 0$$

← Substituted since $12-6h+h^2$ is a polynomial (see Theorem 2.1)

$$= 12 - 6(0) + (0)^2$$

$$= 12.$$

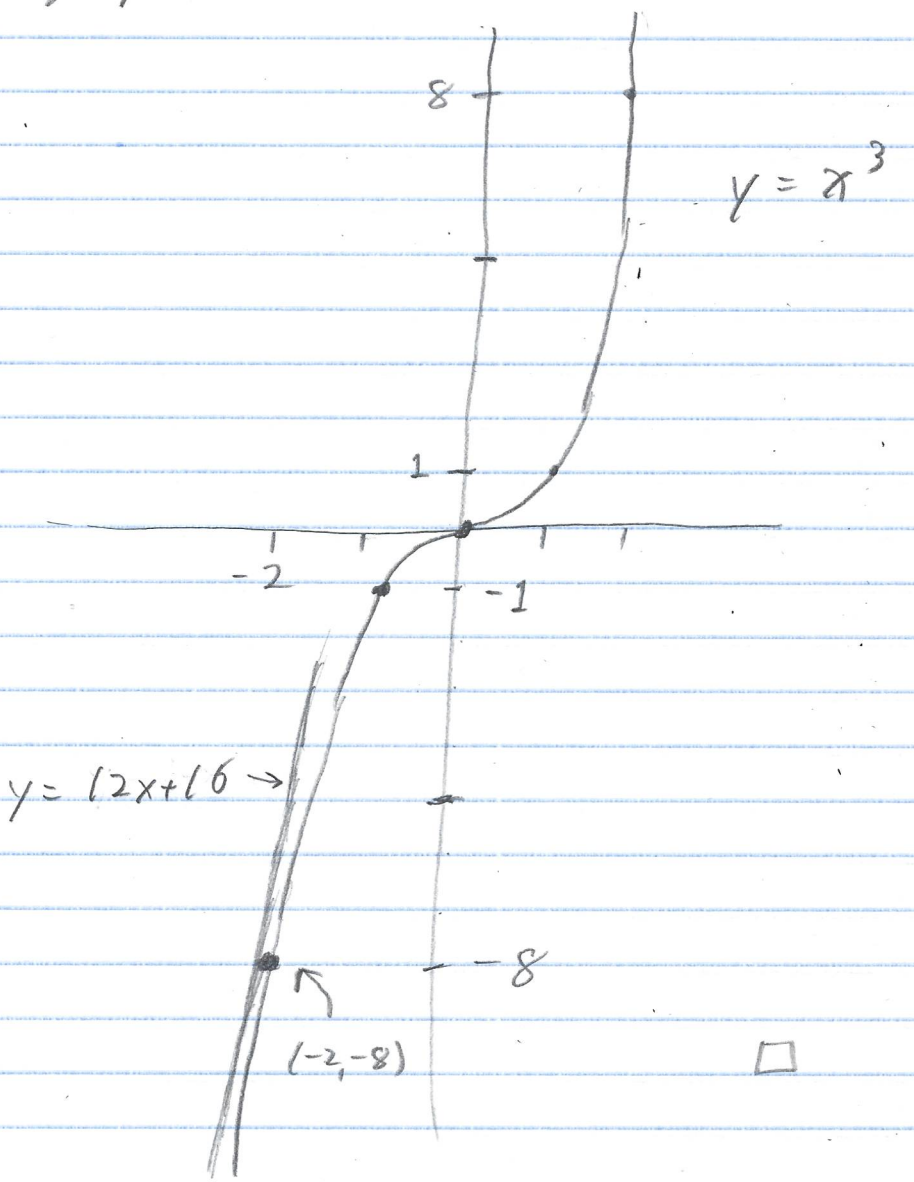
We have $m = 12$ and $(x_1, y_1) = (-2, -8)$. So the desired line is (from the point-slope formula):

$$y - y_1 = m(x - x_1) \text{ or } y - (-8) = (12)(x - (-2))$$

$$\text{or } y + 8 = 12(x + 2) \text{ or } y + 8 = 12x + 24$$

$$\text{or } \boxed{y = 12x + 16.}$$

A graph:



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