3.10.13 For a cylinder of radius r and height h, the volume is $V = \pi r^2 h$.

(a) How is dV/dt related to dh/dt if r is constant?

Solution

With r constant, $V = \pi r^2 h$ is a function of variable h and

$$\frac{dV}{dt} = \frac{d}{dt}\left[\pi r^2 h\right] = \pi r^2 \frac{d}{dt}[h]$$

$$= \boxed{\pi r^2 \frac{dh}{dt} = \frac{dV}{dt}}$$

(b) How is dV/dt related to dr/dt if h is constant?

Solution

With h constant, we have

$$\frac{dV}{dt} = \frac{d}{dt}\left[\pi r^2 h\right] = \pi h \frac{d}{dt}[r^2]$$

$$= \pi h \left[2r\left[\frac{dr}{dt}\right]\right] = \boxed{2\pi h r \frac{dr}{dt}}$$

(c) How is dV/dt related to dr/dt and dh/dt where neither r nor h is constant?

Solution

We need the Product Rule:

$$\frac{d}{dt}[V] = \frac{dV}{dt} = \frac{d}{dt}\left[\pi r^2 h\right]$$

$$= \pi\left(\left[2r\left[\frac{dr}{dt}\right]\right](h) + (r^2)\left[\frac{dh}{dt}\right]\right)$$

$$= \boxed{2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt} = \frac{dV}{dt}}. \square$$